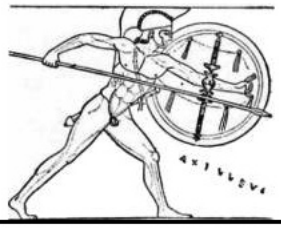
Achilles and the Tortoise sum the geometric series

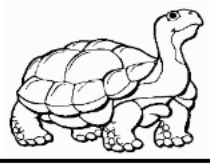
Nathan Pflueger

21 October 2011

This note gives a quick an intuitive explanation for the formula for the sum of a geometric series, based on a version of Zeno's paradox. All images are copied and pasted rather carelessly from the internet.

Achilles, the fleetest of foot of all mortals, is having a footrace with a tortoise. In order to make the race more fair, he gives the tortoise a head start of a units. Of course, Achilles is much faster than the tortoise, so he will catch him eventually. But how long will it take?

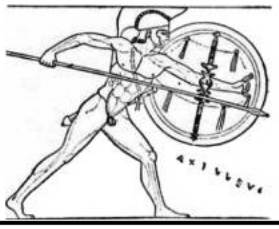


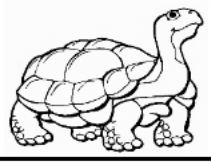


a

To fix notation, assume that Achilles can run at 1 unit per second (to make this realistic, you can imagine that 1 unit is 10 meters), whereas the tortoise can "run" (if you can call it that) at r units per second.

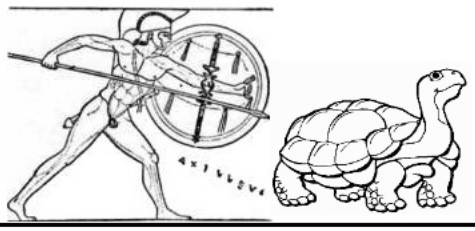
Zeno's paradox argues as follows. In the first a seconds, Achilles reaches the spot where the tortoise started. He is off to a good start. During this time, the tortoise has traveled ar units, so he is still ar units ahead of Achilles.





ar

Now, Achilles will reach the place the tortoise currently stands in another ar seconds. During this time, the tortoise will make a bit more progress, namely ar^2 units, and will thus still be ar^2 units in front of Achilles.



ar^2

The problem is that we can list infinitely many steps this way. Zeno concludes from this that movement is an illusion, because it would be impossible for Achilles to finish an infinite number of these steps, so he will never catch the tortoise.

Another way of looking at things is that Achilles is simply able to do an infinite number of steps in finite time. How much time? The first step takes a seconds, the second step takes ar seconds, the third step takes ar^2 seconds, and so on. So the total amount of time to catch the tortoise should be the sum of all of these.

$$\text{time to catch the tortoise} = a + ar + ar^2 + \cdots \text{ seconds}$$

On the other hand, there is another, much simpler way to compute how long it will take to catch the tortoise. Since Achilles is running at 1 unit per second, and the tortoise is running at r units per second, Achilles is gaining on the tortoise (i.e. shrinking the distance between them) at precisely $1 - r$ units per second. From this we can calculate the amount of time to catch the tortoise very easily.

$$\begin{aligned} \text{tortoise's lead} &= a \text{ units} \\ \text{Achilles gains} & \quad 1 - r \text{ units per second} \\ \text{time to catch the tortoise} &= \frac{a}{1-r} \text{ seconds} \end{aligned}$$

Now, we can equate these two expressions for the time to catch the tortoise, to obtain the formula for the sum of a geometric series.

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}$$