Math 1B, lecture 7: Average value and substitution

Nathan Pflueger

21 September 2011

1 Introduction

This class is divided into two essentially unrelated parts. The first, much shorter, part concerns the average value of a function on an interval. The second part concerns the technique of integration by substitution.

The notion of the average value of a function is one more way to interpret what a definite integral means. The average value of a finite list of numbers is calculated in the familiar way: as the sum of the numbers in the list divided by the length of the list. However, if we want, for example, to speak about the average wind speed in a particular location, we cannot simply take the average of a finite list of numbers. The transition from averages of finite lists to averages of functions is very much analogous to the transition from finite sums to integrals.

The technique of integration by substitution is dual to the chain rule. In differential calculus, the chain rule substantially enlarges the family of functions that can be differentiated exactly, and integration by substitution applies the same technique in reverse to enlarge the family of functions with antiderivatives. Unlike the chain rule, however, there is no simple rule for choosing how to carry out integration by substitution; it is a tool which may apply in many different ways, and the only way to see which ways will work is to attempt them.

Substitution typically *transforms* the integral, thus building analogies between different sorts of integrals. Of course, this analogy is most useful when the transformation ends in an integral that can already be computing by some means, but many transformations produce integrals that are just as difficult as the original. With experience, however, one begins to develop an intuition for how it can be used. Part of what makes substitution fun is that integration becomes a sandbox, with many things to try, and many conceptual links to be found. We will consider some examples here; there will be more examples on the assignment.

One class of functions that can be integrated more or less mechanically with knowledge of integration by substitution are so-called trigonometric polynomials. We will consider these in the last part of the lecture.

The reading for today consists of the following four sections from Gottlieb: §24.2 (average value), §25.2, §25.3 (substitution), and §29.2 (trigonometric substitution). The last of these four sections covers somewhat more material than will concern us in this course. The homework assignment is Problem Set 5, and a topic outline. Notice that this includes weekly problems 3 and 4.

2 Average value

The notion of the average value of a function in calculus really just boils down to a definition.

Definition 2.1. The *average value* of a function $f(x)$ on an interval $[a, b]$ is the value $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$.

This definition is meant to be analogous to the usual definition of the average of a list of numbers: the average is $\frac{1}{\text{(number of elements)}} \cdot \text{(sum of elements)}$. The continuos analog of the number of elements is the length of the interval (it measures "how many values" there are), and the continuous analog of the sum of the elements is the integral of the values of the function.

This definition may be taken as the sole sense of "average value" that you use, on exams for example. The rest of this section is meant solely as motivation.

Another way to view this definition of the average is by the following thought process. Suppose that there is a function $f(t)$ which tells the temperature of a room at a time t, where t ranges in the interval $[0, 24]$ and tells the number of hours that have passed since some reference point. How would we desired the "average temperature" of the room? One option is to pick a sequence of evenly spaced points, measure the temperature at each, and average the values. This amounts to the following calculation. Here $x_1, \ldots x_k$ denote, as usual, n evenly spaced values in the interval $[0, 24]$.

$$\text{(avg. temperature)} \quad \approx \quad \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

We should expect that the larger the value of n, the closer we will approach, by this calculation, the "true" average value of the function. The limit, then, should be the true average. But taking this limit should look quite familiar.

$$\begin{aligned}
\text{(avg. value)} \quad &= \quad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_k) \\
&= \quad \lim_{n\to\infty} \frac{1}{b-a}\frac{b-a}{n}\sum_{k=1}^{n} f(x_k) \\
&= \quad \frac{1}{b-a}\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\frac{b-a}{n} \\
&= \quad \frac{1}{b-a}\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x \\
&= \quad \frac{1}{b-a}\int_a^b f(x)dx
\end{aligned}$$

2.1 Probabilistic interpretation

Where the notion of average value really matters is in the field of probability and statistics. Here is how to interpret the notion of average value in this context. We will not use this interpretation in this course; I mention it only because, for many of you, statistics is the realm in which calculus will be most important to you in your future work.

The average value of $f(x)$ on the interval $[a, b]$ is the expected value of $f(x)$, when x is chosen uniformly at random from the interval $[a, b]$.

The word "uniformly" is important here. It means essentially that every part of the interval $[a, b]$ is just as likely to be drawn from as every other point. More precisely, the probability that the value x is chosen from some given subinterval is proportional to the length of that subinterval.

Of course, I should not that this interpretation is begging the question: I have pushed the burden of definition from the word "average" to the term "expected value." I do this merely to show the analogy; at least for me, the expected value is a more intuitive idea.

The shift from discrete to continuous is fundamental in statistics: basic probability generally concerns probabilities of events chosen from a finite list (e.g. the number shown on a die), but real-world applications are generally continuous (e.g. the height of a member of some population). Hence it is necessary to understand concepts such as "average" in the context of continuous variables. This is the simplest example of such analysis.

To make this idea somewhat more precise, here is a a reformulation of the interpretation I gave above.

If a sequence of n values of x are chosen uniformly at random, and the values $f(x)$ are averaged, then the average will tend to a limit as n goest to infinity; this limit is the average value.

I am being a little dishonest here: in probability, nothing happens for certain, least of all the convergence to a limit. But rest assured that there are ways to make this precise. Here is one of them. Do not worry about parsing this statement if it seems too abstract; I merely include it in case some of you are interested.

(Technical statement) Let ϵ_1, ϵ_2 be any positive numbers (as small as you like). Then for n sufficiently large, the probability that the average of $f(x)$, for n values of x chosen uniformly at random from $[a, b]$, differs from the average value (as defined above) by more than ϵ_1 is less than ϵ_2.

In somewhat less technical language: the average of n values will most likely tend to a limit, in the sense that, by making n large enough, the error will be as small as you wish, at least with probability as large as you wish (but not 1).

As an advertisement: in the spring semester, I will be a TA for the course "Fat Chance," a general education course on probability and statistics that will be taught by my wonderful adviser, Joe Harris. I suggest that all of you tell all of your friends to take it. For more convincing, you should consult the unassailable scores the course received last year on the Q.

3 Substitution

Integration by substitution is the chain rule in reverse.

Example 3.1. Compute the integral $\int_0^2 xe^{x^2}dx$. Here is a lucky guess. Notice that $\frac{d}{dx}e^{x^2} = 2xe^{x^2}$, by the chain rule. Therefore, we can detect the antiderivative of xe^{x^2}: it is $\frac{1}{2}e^{x^2}$. Therefore:

$$\begin{aligned} \int_0^2 xe^{x^2}dx &= \frac{1}{2}e^{x^2}\Big|_0^2 \\ &= \frac{1}{2}e^4 - \frac{1}{2} \end{aligned}$$

What made this example is that we found the antiderivative by cooking up a function whole derivative was the integrand, by means of the chain rule. Let us formalize this process.

The chain rule states that the derivative of a composite function $F(x) = G(u(x))$ is given by $F'(x) = G'(u(x))u'(x)$. The usual way to remember this is that $G'(u(x))$ tells how quickly the function G changes as its input changes near $u(x)$, while $u'(x)$ tells how fast the input of G is changing. In order to reverse this process, we must guess a function $u(x)$ (traditionally, the argument is omitted, and we simply write u, rather than $u(x)$) that will be the "inner" function of a composite function. In this context, this inner function is $u(x) = x^2$. Next, we will attempt to rewrite the integral entirely as a function of u multiplied by $u'(x)$. In symbols:

$$\begin{aligned} \text{if} \quad & f(x) = g(u(x))u'(x) \\ \text{then} \quad & \int f(x)dx = G(u(x)) + C \\ \text{where} \quad & \int g(x)dx = G(x) + C \end{aligned}$$

There is a standard shorthand for this process using differential notation: noticing that $\frac{du}{dx} = u'(x)$, we write $du = u'(x)dx$. This is a bit of a lie, since $\frac{du}{dx}$ is not a fraction, and the symbol du does not refers to a mathematical object that we have carefully defined [1]. Therefore, for our purposes, simply regard the symbol du, where it appears, as *shorthand* for the symbols $u'(x)dx$. Using this shorthand, and suppressing the argument to $u(x)$, we can reformulate the above in the way we will usually use it.

$$\begin{aligned}
\text{let} \qquad & du = u'(x)dx. \\
\text{if} \qquad & f(x)dx = g(u)du \\
\text{then} \qquad & \int f(x)dx = \int g(u)du.
\end{aligned}$$

Here the last integral is meant to be an integral with respect to the variable u; upon commuting an antiderivative $G(u) + C$, we need simply replace the u with whatever function of x it stands for.

Let us illustrate this notation by calculating the same integral a second time.

Example 3.2. Compute $\displaystyle\int_0^2 xe^{x^2}dx$, using the notation above.

To begin, we must choose an appropriate substitution function u. A good first guess is usually the "innermost" function in the integrand; in this case this is $u = x^2$. Then $u'(x) = 2x$. Therefore we compute the indefinite integral as follows.

$$\begin{aligned}
\int xe^{x^2}dx & \qquad \text{substitute } u = x^2, du = 2xdx \\
\int xe^{x^2}dx &= \int xe^u \frac{1}{2x} du \\
&= \frac{1}{2}\int e^u du \\
&= \frac{1}{2}e^u + C \\
&= \frac{1}{2}e^{x^2} + C
\end{aligned}$$

Plugging in the endpoints 0 and 2 for x will then compute the definite integral. Alternatively, we can track the endpoints through the process. There are two usual notations for this, shown side-by-side below.

$$\begin{aligned}
\int_0^2 xe^{x^2}dx & \qquad \text{substitute } u = x^2, du = 2xdx \\
\int_0^2 xe^{x^2}dx &= \int_{x=0}^{x=2} \frac{1}{2}e^u du \\
&= \tfrac{1}{2}e^u\Big|_{x=0}^{x=2} \\
&= \tfrac{1}{2}e^{x^2}\Big|_0^2 \\
&= \tfrac{1}{2}e^4 - \tfrac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\int_0^2 xe^{x^2}dx & \qquad \text{substitute } u = x^2, du = 2xdx \\
\int_0^2 xe^{x^2}dx &= \int_0^4 \frac{1}{2}e^u du \\
& \tfrac{1}{2}e^u\Big|_0^4 \\
& \tfrac{1}{2}e^4 - \tfrac{1}{2}
\end{aligned}$$

Note that the only difference between these two is whether the endpoints are left as values of x or converted immediately into values of u. In the first case, the antiderivative must be turned into a function of x by replacing u by x^2 again in the end. In the second case, the antiderivative can be left in terms of u, since we are feeding it values of u anyway. Both notations are equally acceptable.

---

[1] In differential geometry, such symbols are rigorously defined, and manipulated in various ways. They are called *differential forms*. We will certainly not cover anything about these in this course.

Regarding the notational distinction above: when specifying the limits of integration, it is always unambiguous if you directly specify which variable you are specifying the endpoints in (for example, you can write $\int_{x=0}^{x=2} xe^{x^2}dx$). When the name of the variable is omitted (e.g. in $\int_0^2 xe^{x^2}dx$), it is assumed to be the variable whose differential appears in the integral (in this case, x). In the case of an expression such as $\frac{1}{2}e^u\Big|_0^4$, we can infer from context the the variable going from 0 to 4 is u, since that is the only variable present in the expression, but in any situation where confusion is possible, it will not hurt to fully specify it, e.g. by writing $\frac{1}{2}e^u\Big|_{u=0}^{u=4}$.

The rest of this section will illustrate the technique of substitution using several examples.

Example 3.3. Evaluate $\int \sin x \cdot e^{\cos x}dx$.
The standard attack of substituting for the innermost function works here.

$$\begin{aligned}
\int \sin x e^{\cos x} dx & \qquad u = \cos x,\ du = -\sin x dx \\
&= \int (-e^u)du \\
&= -e^u + C \\
&= -e^{\cos x} + C
\end{aligned}$$

Example 3.4. Evaluate $\int_0^3 x\sqrt{x+1}dx$.

Same trick.

$$\begin{aligned}
\int_0^3 x\sqrt{x+1} & \qquad u = x+1,\ du = dx \\
&= \int_1^4 x\sqrt{u}du \\
&= \int_1^4 (u-1)\sqrt{u}du \\
&= \int_1^4 (u^{3/2} - u^{1/2})du \\
&= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\Big|_1^4 \\
&= \left(\frac{2}{5}\cdot 32 - \frac{2}{3}\cdot 8\right) - \left(\frac{2}{5} - \frac{2}{3}\right) \\
&= \left(\frac{64}{5} - \frac{16}{3}\right) - \left(-\frac{4}{15}\right) \\
&= \frac{112}{15} + \frac{4}{15} \\
&= \frac{116}{15}
\end{aligned}$$

Example 3.5. Evaluate $\int \cos(\sqrt{x})dx$.
Try the usual trick of substituting for the innermost function.

$$\begin{aligned}
\int \cos(\sqrt{x})dx & \qquad u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}dx \\
&= \int \cos(u)\cdot 2\sqrt{x}du \\
&= \int u \cos u du
\end{aligned}$$

There are two points to be made here. The first is that the integral passed through an intermediate stage in which the integrand contained both u and x. This is perfectly acceptable. Indeed, bearing in mind that u is really shorthand for a function $u(x)$ of x, and du is just shorthand for $u'(x)dx$, we really have an integral purely in terms of x all along. As long as we can transmute everything in the integrand, eventually, into functions of u, we will have achieved a valid transformation of the integral.

The second point may already be itching at you: we are not finished! Indeed, it is not clear that we have made our lives any easier by means of this transformation, since we do not currently have a technique to evaluate the integral of $u \cos u$. This is often the case: there is no guarantee that transforming the integral will make it easier to evaluate. However, this is not always a tragedy: a transformation, at the very least, establishes an analogy. It is better to understand that two instances of ignorance are the same type of ignorance than to merely be ignorant about both of them.

In this case, however, we will have a technique for this new integral as soon as the next class, when this integral will make a reappearance.

We proceed now to a particular familiar of integrals that can be solved more or less completely using the present techniques.

4 Trigonometric polynomials

The techniques at hand allow us to integrate all so-called *trigonometric polynomials*; these are functions such as $f(x) = \sin^2 x \cos^7 y - \sin x \cos^2 x$ built up as polynomials of the functions $\sin x$ and $\cos x$. The basic tools will be integration by substitution as outlined above, and some strategic use of two key trigonometric identities.

$$\begin{aligned} 1 &= \sin^2 x + \cos^2 x && \text{(The Pythagorean theorem)} \\ \cos(2x) &= \cos^2 x - \sin^2 x && \text{(The double-angle formula for cosine)} \end{aligned}$$

It is important to notice that these two identities can be fruitfully combined to obtain two other common statements of the double angle formula for cosine.

$$\begin{aligned} \cos(2x) &= 2\cos^2 x - 1 \\ \cos(2x) &= 1 - 2\sin^2 x \end{aligned}$$

It is worth noting that these two formulas, taken together, imply the original two formulas above.

There are many other trigonometric identities, some of which are discussed in Gottlieb, §29.2, and are used for a more diverse family of integrals. Personally, I have always found it a bit morally repugnant to memorize identities such as these, despite the fact that many curricula make such memorization necessary. Nonetheless, these two basic formulas are so central in calculus that it really is prudent to commit them to memory. I provide geometric arguments for both of these statements in an appendix (which is absolutely not required for this course, and mainly included for my own sense of completeness).

One important lesson to take from trigonometric identities is that while $\sin x$ and $\cos x$ (as well as their cousins, $\cos 2x$ and so forth) are in some sense different functions, but they are deeply bound together in ways that make them behave in tandem. The aim of trigonometric identities is to grasp the puppet master manipulating both functions and shake him until he gives in to our demands.

We will illustrate how to evaluate integrals of trigonometric polynomials by means of some examples.

Example 4.1. Evaluate $\displaystyle\int \sin^2 x \cos x dx$. Observe that $\cos x dx = d(\sin x)$. Hence we can solve this be the substitution $u = \sin x$. The integral becomes $\int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C$.

Example 4.2. Evaluate $\displaystyle\int \sin^3 x \cos^2 x dx$.

Flush with our previous victory, we can attempt again the substitution $u = \sin x$. This results in the integral $\int u^3 \cos x du$. Unfortunately, $\cos x$ is not yet in terms of u, and there is not a convention way to put it in this form (one option would be $\sqrt{1-u^2}$, but this presents two problems: it only works in the domain where $\cos x \geq 0$, which can be surmounted, and it does not make the integral particularly easier).

Try instead the substitution $u = \cos x$. Then $du = -\sin x dx$, and gives the integral $-\int \sin^2 x \cdot u^2 du$. In this case, we actually can express the pesky $\sin x$ term in terms of u by means of the Pythagorean theorem. The full computation is shown below.

$$
\begin{aligned}
\int \sin^3 x \cos^2 x dx \qquad & u = \cos x,\ du = -\sin x dx \\
= \quad & -\int \sin^2 x u^2 du \\
= \quad & -\int (1-u^2)u^2 du \\
= \quad & \int (u^2-1)u^2 du \\
= \quad & \int (u^4-u^2) du \\
= \quad & \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
= \quad & \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C
\end{aligned}
$$

Notice that what made the substitution $u = \sin x$ work in the first example was precisely that the exponent of $\cos x$ was odd, while what made the substitution $u = \cos x$ work in the second example was precisely that the exponent of $\sin x$ was odd.

Upon some reflection, the reader will become convinced from these two examples that a substitution of $u = \sin x$ or $u = \cos x$, combined with an application of the Pythagorean theorem, will transform $\int \sin^a x \cos^b x dx$ into the integral of an ordinary polynomial, and thus give a way to the compute the integral, in all cases where either a is odd or b is odd. What can be done in case both exponents are even? The following example shows what must be done in the simplest possible case.

Example 4.3. Evaluate $\displaystyle\int \sin^2 x dx$.

In this case, both substitutions $u = \sin x$ and $u = \cos x$ will not particularly help. What is needed is to apply the double angle formula for cosine: since $\cos(2x) = 1 - 2\sin^2 x$, it follows that $\sin^2 x = \frac{1}{2}(1-\cos(2x))$. Therefore, the integral can be computed as follows.

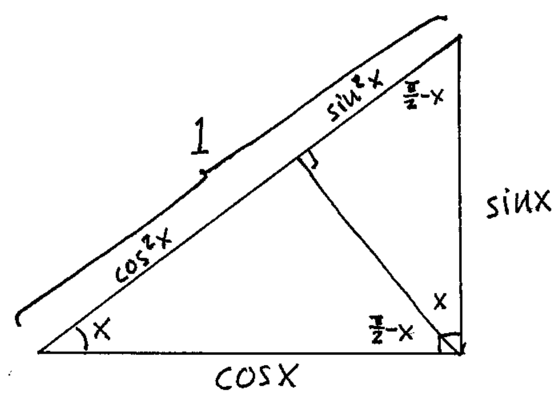
$$
\begin{aligned}
\int \sin^2 x dx &= \frac{1}{2}\int (1-\cos(2x))dx \\
& \quad u = 2x,\ du = 2dx \\
&= \frac{1}{2}\int (1-\cos u)\frac{1}{2}du \\
&= \frac{1}{4}\int (1-\cos u)du \\
&= \frac{1}{4}(u - \sin u) + C \\
&= \frac{1}{4}(2x - \sin(2x)) + C \\
&= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C
\end{aligned}
$$

In general, a combination of these two techniques will work. We will not describe the whole recipe in painful detail.

4.1 Appendix: Trigonometric identities

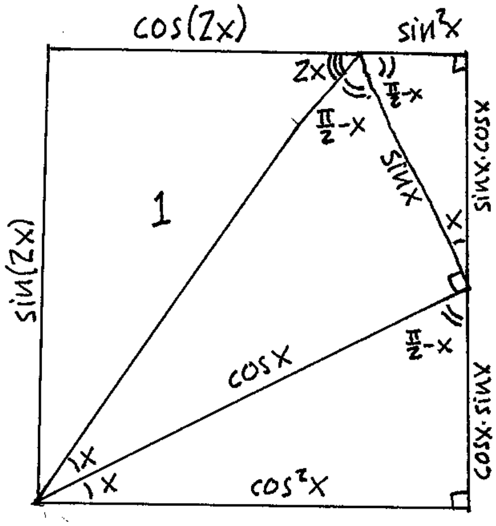
This subsection is very much not required for the course.

The following shows how I picture the Pythagorean theorem (which is equivalent to the identity $\sin^2 x + \cos^2 x = 1$.



Depicted is a right triangle with hypotenuse 1, and angle x in one corner. Its legs, thus, have length $\sin x$ and $\cos x$ (by definition). Drop a perpendicular segment to the hypotenuse from the opposite vertex. This dissects the triangle into two similar triangles, with hypotenuses $\sin x$ and $\cos x$. Then calculating the lengths of their edges, we find that we have split the hypotenuse of the original triangle into two segments, whose lengths are $\cos^2 x$ and $\sin^2 x$. Hence $1 = \sin^2 x + \cos^2 x$.

The following picture shows how I picture the double-angle formula for cosine.



Essentially, two right triangles with angle x are stacked on top of each other, the hypotenuse of one becoming the leg of the other. If the hypotenuse of the top triangle is 1, then its leg BD is $\cos x$. This being the hypotenuse of the bottom triangle, the bottom triangle has leg $\cos x \cdot \cos x = \cos^2 x$. But now, enclosing the whole figure in a rectangle, another right triangle with angle x occurs in the upper-right corner, which produces an edge of length $\sin^2 x$ in a similar manner. On the other hand, a right triangle is formed in the upper left, one of whose angles is $2x$, with hypotenuse 1, so its legs are $\sin(2x)$ and $\cos(2x)$. Using the face that this whole figure forms a rectangle, the following two double-angle formulas are visible.

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ \sin(2x) &= 2 \sin x \cos x \end{aligned}$$

Note that we could have just as easily have stacked two triangle with two different angle x and y, from which we would have obtained the more general angle sum formulas for sine and cosine.

To end the appendix, I will mention the more mature way to conceive of this situation (and in fact all of trigonometry). A remarkable fact (which will come out later in the course) is the following formula, which relates trigonometric functions and exponential functions via complex numbers. Here i is a square root of -1.

$$e^{ix} = \cos x + i \sin x$$

Of course, one could object that this isn't a fact so much as a definition. However, what makes the definition have content is that the exponential function, defined on imaginary numbers in this way, retains its defining property: that $e^a \cdot e^b = e^{a+b}$.

In particular, both of the trigonometric identities that we use in this class follow immediately from this formula, as follows:

$$
\begin{aligned}
1 &= e^{ix} \cdot e^{-ix} \\
&= (\cos x + i \sin x) \cdot (\cos x - i \sin x) \text{ (since } \cos(-x) = \cos x \text{ and } \sin(-x) = -\sin x) \\
&= \cos^2 x + \sin^2 x
\end{aligned}
$$

$$
\begin{aligned}
\cos(2x) + i \sin 2x &= e^{2ix} \\
&= e^{ix} \cdot e^{ix} \\
&= (\cos x + i \sin x) \cdot (\cos x + i \sin x) \\
&= (\cos^2 x - \sin^2 x) + i(2 \sin x \cos x)
\end{aligned}
$$