Math 1B, lecture 5: area and volume

Nathan Pflueger

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1 Introduction

This lecture and the next will be concerned with the computation of areas of regions in the plane, and volumes of regions in space. The analysis will be very much analogous to the computation of total mass from density that we considered last week. The central technique is the same: to find the whole, slice it into many parts, approximate each part, and sum them. If the error of this scheme shrinks as the number of slices grows, then the whole (mass, area, or volume) will be expressed by an integral. The goal of these classes will be to describe how to determine the right integrals to expresses various sorts of areas and volumes.

A central feature of these two classes will be the observation that *there may be more than one way to slice the whole.* Sometimes, one way is better than another; in many cases the key to a problem is changing the manner of slicing. This very basic idea is at the core of many important ideas in mathematics; we will encounter it again in our study of the integration by parts technique.

The reading for today is Gottlieb §27.2 and the first part of §28.1 (up to page 856). The homework is Problem Set 3. You should begin working on weekly problem 4, although it is not due until problem set 5.

2 Guiding examples

The following examples will guide the discussion in this lecture. We may not cover all of them in class. A full solution to each will be given at the appropriate place in the notes. Brief answers to all of these examples are provided at the end of these notes for reference.

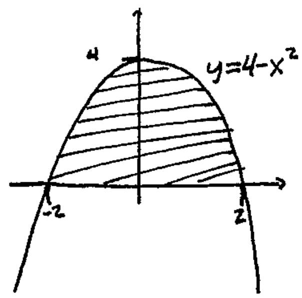
1. (a) Find the area of the region between the curve $y = 4 - x^2$ and the x-axis.
   (b) Find the area between the curves $y = x^2$ and $y = 2 - x^2$.
2. Find the area of the region enclosed by the curves $y = \sqrt{x}$, $y = -\sqrt{x}$, and $x = 4$.
3. Evaluate $\displaystyle\int_0^1 \sin^{-1}(x)dx$.
4. (a) Evaluate $\displaystyle\int_1^e \ln x dx$.
   (b) Find the antiderivative of $\ln x$.
5. Suppose you have forgotten the formula for the area of a circle, but you remember that the perimeter of a circle of radius r is $2\pi r$. How could you recover the formula for the area?
6. A coffee cup has diameter 3 inches at the top, diameter 2 inches at the bottom, and is 5 inches tall. Find the volume of the cup.
7. Find the formula for the volume of a sphere.
8. Find the formula for the surface area of a sphere.

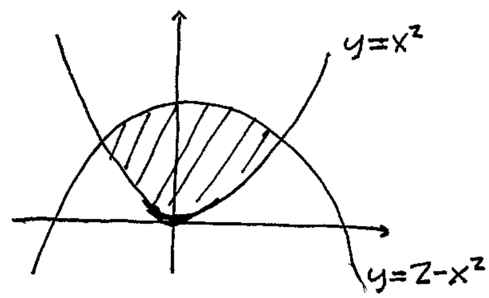
3 Areas in the plane by slicing

Compute the area between these two curves. This is perhaps the quintessential AP Calculus problem: easy to pose, not too difficult to solve, and apparently totally abstract and meaningless. Although this last criticism may well be the case, the problem still does illustrate some important ideas, so let's consider it. It will lead naturally into a more general discussion of slicing to find area.

3.1 The area between two curves

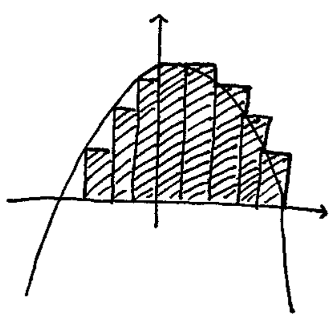
Consider guiding example 1. The two regions are sketched below.





Notice that the curve $y = 4 - x^2$ meets the x axis at $x = -2$ and $x = 2$. So, by one of the usual definitions of the definite integral, the area under it is $\int_{-2}^{2}(4 - x^2)dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = (8 - \frac{8}{3}) - (-8 + \frac{8}{3}) = \frac{32}{3}$ [1].

Now, examine this calculation in terms of the slicing implicit in the integral sign. Essentially, we are chopping this segment into n thin pieces. If the endpoints of these rectangles are $x_0, x_1, \ldots, x_n$ as usual, with the spacing given by $\Delta x = 4/n$, then the area of the k^{th} piece is approximates by the area of a rectangle with width Δx and height[2] $4 - x_k^2$. So what is at work here is that:

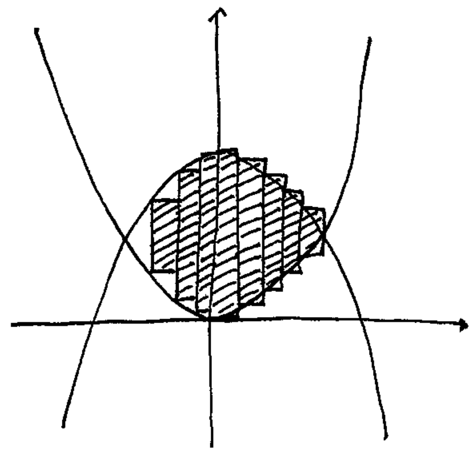


$$\begin{aligned} \text{Area} &\approx \sum_{k=0}^{n}(4 - x_k^2)\Delta x, \\ &= \lim_{n\to\infty}\sum_{k=0}^{n}(4 - x_k^2)\Delta x, \\ &= \int_{-2}^{2}(4 - x^2)dx. \end{aligned}$$

So much for part (a). Part (b), then, is essentially similar. Now, the two endpoints are the places where the two curves meet, i.e. where $x^2 = 2 - x^2$, which are given by $x = -1$ and $x = 1$. The area of this region can similarly be approximated by slicing it into vertical rectangles.

[1] It may be amusing to note that this area is also quickly calculated by using Simpson's rule, this function is quadratic.

[2] The height could also be chosen as $4 - x_{k-1}^2$ for the left sum, or their average for the trapezoid sum, and so on. There are many approximations to use; this was the subject of the last two lectures. The point is that they all converge to the same integral, which is what interests us in this lecture.



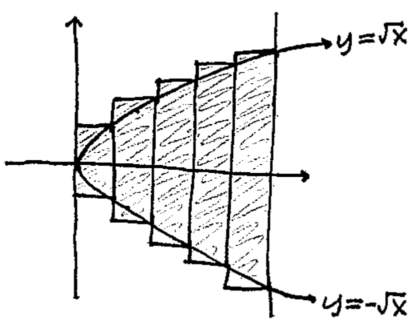
Now the height of the k^{th} rectangle is can be taken to be the difference between the two y values, namely $(2 - x_k^2) - x_k^2 = 2 - 2x_k^2$. Thus we can similarly express this area as an integral.

$$\begin{aligned} \text{Area} &\approx \sum_{k=0}^{n} (2 - 2x_k^2)\Delta x, \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} (2 - 2x_k^2)\Delta x, \\ &= \int_{-1}^{1} (2 - 2x^2)dx. \end{aligned}$$

This integral evaluates to $\frac{8}{3}$. So the answers to guiding example 1 are (a) $\frac{32}{3}$, and (b) $\frac{8}{3}$.

3.2 Reslicing integrals

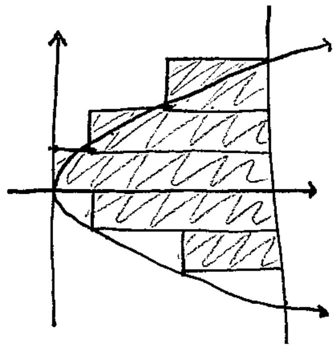
Consider guiding example 2. We can use the same technique as in part (b) of guiding example 1: slice the region vertically, and form an integral.



$$\text{Area} = \int_0^4 (\sqrt{x} - (-\sqrt{x}))dx$$

Of course, this integral is the same as $2\int_0^4 \sqrt{x}dx$, which can be calculated to be $\frac{32}{3}$.

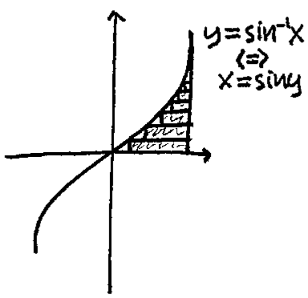
However, observe that we could just as easily have sliced this region *horizontally* instead of vertically. Now the variable of integration is y instead of x. The area we seek is the area between the line $x = 4$, and one curve that can be expressed by the equation $x = y^2$. Therefore, we can also do the computation in the following way.



$$(\text{Area}) = \int_{-2}^{2} (4 - y^2)dy$$

This integral can then be evaluated to be $\frac{32}{3}$. Fortunately, this shatters none of our previously held beliefs.

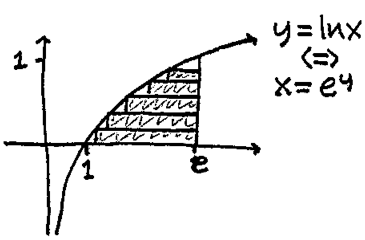
In this case, the fact that the area could be sliced in two different ways was amusing, but not especially useful: we only need one method to find the correct answer. But now consider guiding example 3. Here, none of the methods currently available will suffice to compute this integral as it stands. However, consider what happens if we slice horizontally, instead of vertically. Now, each slice is the area between the line $x = 1$ and the curve $x = \sin^{-1} y$. The coordinate y varies from 0 to $\pi/2$. So we can write the same integral in a very different way.



$$\int_0^1 \sin^{-1} x dx = \int_0^{\pi/2} (1 - \sin y)dy$$

Now this integral is something that we can already handle: its value is $[y + \cos y]_0^{\pi/2} = (\pi/2+0) - (0+1) = \frac{\pi}{2} - 1$. This is the answer to guiding example 3.

Guiding example 4(a) can be done with the same technique: the desired area is bounded by the line $x = e$ and the curve $x = e^y$.



$$\int_1^e \ln x dx = \int_0^1 (e - e^y)dy$$

The value of this integral is $[ey - e^y]_0^1 = (e - e) - (0 - 1) = 1$. Part (b) follows precisely the same reasoning; simply replace the e in the limits of integration by a variable t. The same re-slicing gives the following.

$$\begin{aligned}\int_1^t \ln x dx &= \int_0^{\ln t} (t - e^y) dy \\ &= [ty - e^y]_0^{\ln t} \\ &= (t \ln t - t) - (0 - 1) \\ &= t \ln t - t + 1\end{aligned}$$

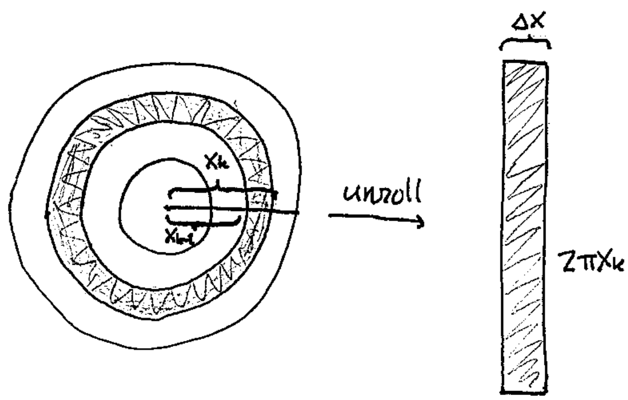
Therefore, we have determined that the antiderivative of $f(x) = \ln x$ is $x \ln x - x + C$ (recall that an antiderivative is well-defined only up to addition of a constant; hence the constant 1 that appeared in the equation above can be disregarded).

Notice that this same technique will also work to find antiderivatives of many inverse functions. In fact, this technique is essentially equivalent to a special case of the technique of *integration by parts*, which we will encounter soon.

3.3 Three ways to slice a circle

So far we have computed the areas of some regions by slicing horizontally or vertically, but there are certainly other ways to slice. Another way is to slice *radially*, which provides a neat solution to guiding example 5.

Suppose that all we know is that the circumference of a circle with radius r is $2\pi r$. This should not be hard to swallow; in fact, it is the usual definition of π as the ratio of circumference to diameter. So let's use this knowledge to slice the circle into pieces that we can at least approximate: cut it into n concentric rings of increasing radius.



Now, the k^{th} ring has inner radius x_{k-1} and outer radius r_k (where $x_0 = 0$, $x_n = r$, and $x_1, x_2, \ldots x_{n-1}$ are evenly spaced between these two). The thickness of the ring is Δx. We could in principle find the area of the ring by subtracting the area of the circle inside from the area of the circle outside, but this reasoning would obviously be circular[3]. So argue on fuzzier grounds: if this ring were cut and straightened out, it would be nearly a rectangle of width Δx and height somewhere between $2\pi x_{k-1}$ and $2\pi x_k$ (the circumferences of the two circles bounding the ring). This can be approximated by $2\pi x_k$, since for large n, x_k and x_{k-1} will be essentially equal[4]. Now, we can write this as a sum, let n got to ∞, and obtain an integral.

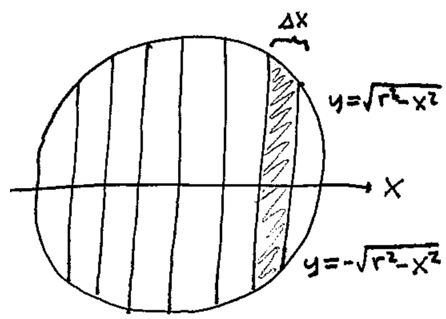
[3]lol

[4]In some sense, the fact that we can choose between x_k and x_{k-1}, or anything in between, comes from the fact that both left-approximation and right-approximation both converge to the same value of the integral.

$$
\begin{aligned}
(\text{Area of circle}) &\approx \sum_{k=1}^{n} 2\pi x_k \Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_k \Delta x \\
&= \int_0^r 2\pi x dx \\
&= \pi r^2
\end{aligned}
$$

Now, notice something striking about this: what we have done, essentially, is show that the antiderivative of the function $C(r) = 2\pi r$ giving the circumference of a circle is (up to a constant) the function giving the area of a circle. another way: *the function $C(r)$ for the circumference of a circle is the derivative of the function $A(r)$ for the area of a circle.* It is worth thinking carefully about why this should not surprise you.

This observation constitutes the answer to guiding example 5. As the subsection title suggests, there are other ways to obtain the area of a circle. One is to slice it vertically (or horizontally, really; it will look the same).

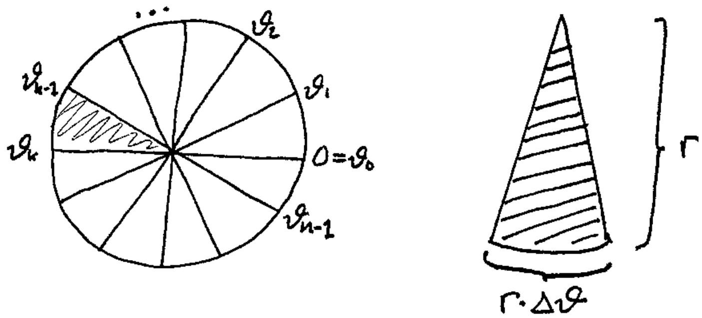


Turing this slicing scheme into an integral as in the previous section, this suggests that the circle's area should be

$$\int_{-r}^{r} 2\sqrt{r^2 - x^2} dx.$$

This will also, of course, give the result πr^2, but it is somewhat more difficult. The reason is that this integral, unlike the integral that we obtained by slicing radially, is rather difficult to compute. We will see a method for computing it later in the course. For the impatient, I simply remark that the relevant substitution is $u = \cos^{-1}(x)$ and warn that the road ahead is fraught with peril.

The third way of slicing is what might be termed *polar* slicing: slice the circle into n equal pie wedges.



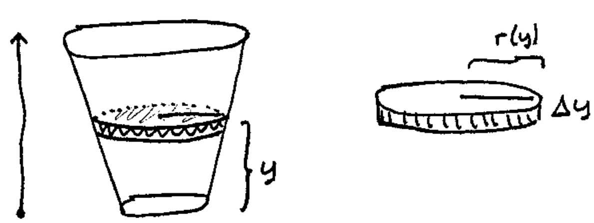
Now, the variable of integration will be θ, the angle made by the line to a point with the x axis. So $\theta_0 = 0$, $\theta_n = 2\pi$, and the other θ_k are evenly spaced. Observe that each slice is shaped like a pie wedge, with angle $\Delta\theta$ at the vertex and radius r. The length of the curved part will be $rDelta\theta$. Now observe that for large n, the curved side will be nearly flat, so we can approximate the area of this pie wedge by the area of a triangle with base $r\delta\theta$ and height r. The area of such a triangle is $\frac{1}{2}(r\Delta\theta)r = \frac{1}{2}r^2\Delta\theta$. Adding these up, we see that the area of the circle should be the following.

$$\begin{aligned}
\text{(Area of circle)} &\approx \sum_{k=1}^{n} \frac{1}{2} r^2 \Delta\theta \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2} r^2 \Delta\theta \\
&= \int_0^{2\pi} \frac{1}{2} r^2 d\theta \\
&= 2\pi \cdot \frac{1}{2} r^2 \\
&= \pi r^2
\end{aligned}$$

Thus this third way of slicing also gives the correct area of the circle.

4 Volumes in space by slicing

Solid objects can be sliced to. For example, consider guiding example 6. The coffee cup could be sliced any number of ways, and many of them will give a good way to compute the volume. We will focus here on one way of slicing: make horizontal slices, thus dissecting the inside of the cup into circles.



Now we need to define a variable. Let y be the distance from the bottom of the cup. The volume of each slice will be roughly equal to the volume of a cylinder with height Δy. What should the radius of the slice be? If we imagine slicing the cup at distance y, this gives a radius that is somewhere between 1 and 1.5. Call the radius at height y $r(y)$. Then $r(0) = 1$ and $r(5) = 1.5$. Furthermore, $r(y)$ is a linear function. The slope of $r(y)$ must be $\frac{1.5-1}{5} = \frac{1}{10}$. Thus the function must be $r(y) = 1 + \frac{1}{10}y$.

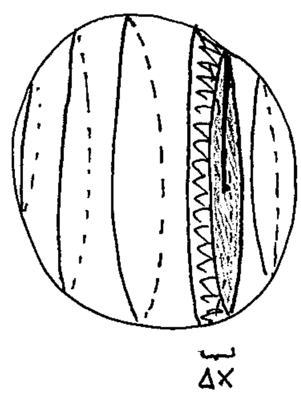
From this, it appears that the volume of a slice cut at height y_k is approximately $\pi(y_k)^2\Delta y = \pi(1 + \frac{1}{10}y_k)^2\Delta y$. Thus we obtain an integral for the volume of the cup.

$$\begin{aligned}(\text{Volume of cup}) &\approx \sum_{k=1}^{n} \pi\left(1+\frac{1}{10}y_k\right)^2 \Delta y \\ &= \int_0^5 \pi\left(1+\frac{1}{10}y\right)^2 dy\end{aligned}$$

The most painless way to evaluate this integral is to perform substitution for $u = 1 + \frac{1}{10}y$. Then $du = \frac{1}{10}dy$, hence $\int_0^5 \pi\left(1+\frac{1}{10}y\right)^2 dy = \int_1^{3/2} \pi u^2 \cdot 10du = \frac{10}{3}\pi((\frac{3}{2})^3 - 1^3) = \frac{95}{12}\pi$. This is the answer to guiding example 6.

4.1 Two ways to slice a sphere

We conclude with a discussion of the volume of a sphere, analogous to the discussion of the area of a circle. How might we slice a sphere in order to compute its area? The simplest way to slice a sphere is into circular slices.



Assume that the circle is placed at the center of the coordinate axes, and is sliced along the x axis. Then the cross-section of a slice at x is a circle. From the Pythagorean theorem, the radius of this circle must be $\sqrt{r^2 - x^2}$, and so its area must be $\pi(r^2 - x^2)$. The thickness of the slice is denoted, as usual, by Δx; so the volume of the k^{th} slice is approximately $(r^2 - x_k^2)\Delta x$. Converting this into an integral as in the previous examples, we obtain the following result.

$$
\begin{aligned}
\text{(Volume of sphere)} &= \int_{-r}^{r} \pi(r^2 - x^2)dx \\
&= \pi \left[r^2 x - \frac{1}{3}x^3 \right]_{-r}^{r} \\
&= \pi(2r^3 - \frac{2}{3}r^3) \\
&= \frac{4}{3}\pi r^3
\end{aligned}
$$

So this is the answer to guiding example 7.

Now, note that we could equally well have sliced the sphere into a series of concentric spherical shells (this is rather difficult to draw). Without belaboring the point, carefully thinking about this will lead to the main insight behind the solution of guiding example 8: *the function S(r) for the surface area of a sphere of radius r is the derivative of the function $V(r)$ for the volume of a sphere of radius r.* In particular, since we know that the volume is given by $\frac{4}{3}\pi r^3$, the surface area must be given by the derivative of this, namely $4\pi r^2$. This is the answer to guiding example 8.

The facts that circumference is the derivative of area and surface area is the derivative of volume generalize to higher-dimensional "spheres" as well, although this is not the place to begin a discussion of what these notions should correspond to for a four-or-more-dimensional sphere.

5 Answers to guiding examples

1. (a) 32/3
   (b) 8/3
2. 32/3
3. $\frac{1}{2}\pi - 1$
4. (a) 1
   (b) $x \ln x - x + C$
5. $\int_0^r (2\pi x)dx = \pi r^2$
6. $\frac{95}{12}\pi$
7. $\frac{4}{3}\pi r^3$
8. $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$