Lecture 31: Second order homogeneous equations II

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1 Introduction

This lecture gives a complete description of all the solutions to any differential equation of the form $x''(t) + bx'(t) = cx(t) = 0$, where b, c are constants.

The basic idea is the same as the last example from the previous lecture: determine all possible exponential solutions by solving a quadratic equation. Complications arise when this quadratic equation (called the characteristic equation) has complex roots, or has only one root. Upon treating these possibilities, a full solution is obtained.

At the heart of the argument is a beautiful mathematical insight known as Euler's formula, which provides a bridge between exponential functions and trigonometric functions, through the use of complex numbers. The two main behaviors that are seen in second order homogeneous equations are oscillation and exponential behavior; Euler's formula shows that these are really two sides of the same coin. It is a vivid testament to the power of imaginative generalization: by introducing a fantasy (imaginary numbers), the picture becomes substantially simpler, and analogies between apparently disparate notions become possible and quite revealing.

We do not have the time to elaborate on these ideas much in this course, but they form the basis for the study of so-called linear systems of differential equations. The final topic of the course will be to study some very simple examples of systems of differential equations (including linear systems), but a fuller treatment must be looked for in a more focused course on differential equations.

The reading for today is Gottlieb 31.6. The homework is problem set 30 and a a topic outline.

2 Statement of the problem

We begin with the same situation as the previous lecture. Suppose that two constants b, c are given, and we wish to solve the following differential equation. Here x is a function of the variable t.

$$x''(t) + bx'(t) + cx(t) = 0$$

This is called a *second-order homogeneous linear equation with constant coefficients.* For today, I will generally refer to this simply as a *second-order homogeneous equation.*

As discussed in the previous lecture, to solve this equation it suffices to find two independent solutions. The general solution will then consist of arbitrary linear combinations of these two independent solutions. Three specific examples were considered. Here are these three examples, with their general solutions. As usual, C_1, C_2 denote two arbitrary constants.

Equation	General solution
(I) $x'' = 0$	$x(t) = C_1 t + C_2$
(II) $x'' + x = 0$	$x(t) = C_1 \sin t + C_2 \cos t$
(III) $x'' + 3x' + 2x = 0$	$x(t) = C_1 e^{-t} + C_2 e^{-2t}$

In all three cases, the general solution was essentially found by guesswork. In this lecture, we explain how to solve all such equations in a systematic way; the most essential types of behavior are displayed in these three examples, as we shall see.

3 The characteristic equation

Recall how we went about solving equation (III): we made a guess that there might be exponential solutions, and then determined what the appropriate exponent should be. In particular, the following was seen to be true: $x(t) = e^{rt}$ solves $x'' + 3x' + 2x = 0$ if and only if $r^2 + 3r + 2 = 0$. There are two such values of r: -2 and -1. Therefore there are two such exponential solutions: e^{-t} and e^{-2t}.

In general, for any second-order homogeneous equation, the quadratic equation describing which values r will give exponential solutions is called the characteristic equation.

Differential equation:	$x''(t) + bx'(t) + cx(t) = 0$
Characteristic equation:	$r^2 + br + c = 0$

In general, we have the following fact: *a number r is a solution to the characteristic equation if and only if the function e^{rt} is a solution to the differential equation.* Therefore the characteristic equation gives a way to find solutions to the differential equation. If it has two real roots, then it identifies all solutions, as in equation (III). As we shall see, however, it determines all solutions in all other cases as well.

The key is to study the solutions, real or otherwise, to the characteristic equation. We can often find these by hand, via factoring. Alternatively, they can always be found with the quadratic formula, as follows.

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

The behavior of the solutions to the differential equation depend on what kind of solutions occur to the characteristic equation. Consider what this looks like in the three main examples that we are considering.

Differential equation	Characteristic equation	Roots of characteristic equation
(I) $x'' = 0$	$r^2 = 0$	$r = 0$ (only one root)
(II) $x'' + x = 0$	$r^2 + 1 = 0$	$r = i, r = -i$
(III) $x'' + 3x' + 2x = 0$	$r^2 + 3r + 2 = 0$	$r = -1, r = -2$

Notice that in case (II), the letter i denotes the imaginary number $\sqrt{-1}$.

These three equations display the three cases which must be understood. Any quadratic equation can either one real root, two reals roots, or two complex roots. We shall consider each case individually.

4 Two real roots

Consider first the situation in equation (III): the characteristic equation has two real roots. In fact, this case is the simplest: each of the two roots gives an exponential solution, and these two solutions together give the general solution.

In symbols: if the characteristic equation has two real roots r_1, r_2, then the general solution of the differential equation is $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

Example 4.1. Consider the following differential equation.

$$x'' - 8x' + 7x = 0$$

The characteristic equation is $r^2 - 8r + 7 = 0$. This can be factored as $(r - 1)(r - 7) = 0$, hence there are two real roots: 1 and 7. Therefore there are two exponential solutions $x(t) = e^t$ and $x(t) = e^{7t}$ to this differential equation, and the general solution is $x(t) = C_1 e^t + C_2 e^{7t}$.

5 Two complex roots

Consider now what happens when the characteristic equation has two complex roots (i.e. roots that are not real numbers). For example, the characteristic equation of equation (II) has roots i and $-i$. As we saw by inspired guessing, two solutions to (II) are $x(t) = \sin t$ and $x(t) = \cos t$. How is this related to the roots of the characteristic equation?

In fact, the relation comes from a rather remarkable insight of Euler[1]. Suppose that we forgot that i was not a real number and tried to claim that $x(t) = e^{it}$ is a solution to $x'' + x = 0$. What could possibly be meant by this? The answer is provided by the following equation, which is referred to as *Euler's formula.*

$$e^{it} = \cos t + i \sin t$$

I explain in an appendix various ways to arrive at this formula. The first argument that students usually see is by Taylor series, although this is not the most revealing argument, in my opinion.

What could possibly be meant by a statement like Euler's formula? In one sense, it is merely a definition. What is accomplishes is that it takes a notion that we have been using frequently in calculus – the notion of the exponential function – and decides what it should mean when it is given an imaginary input. The result will in general be a complex number (i.e. a sum of an imaginary and a real number). What makes this definition meaningful is that it respects the most essential properties of the exponential function.

In particular, observe the following: the real component of e^{it} is $\cos t$, and the imaginary component is $\sin t$. Both of these, as functions, solve the differential equation $x'' + x = 0$. In effect, the function e^{it} is two functions in one (one for the real part, one for the imaginary), and *both* of these functions solve the original differential equation.

Now, given that the "solution" $x(t) = e^{it}$ gave us two real solutions for the price of one, will we get two more solutions for the other root of the characteristic equation, $-i$? Actually, we get the same two solutions. Observe that by Euler's formula:

$$\begin{aligned} e^{-it} &= \cos(-t) + i \sin(-t) \\ &= \cos t - i \sin t \end{aligned}$$

So again we get two solutions for the price of one: they are $x(t) = \cos t$ (the real part) and $x(t) = -\sin t$ (the imaginary part). But essentially these are the same two solutions that we obtained from e^{it}; one of them has simply changed sign.

In fact, Euler's formula gives a way to solve any second-order homogeneous differential equation whose characteristic equation has two complex roots. Consider the following examples.

Example 5.1. Consider the differential equation $x'' + 4x' + 13x = 0$.

The characteristic equation is $r^2 + 4r + 13 = 0$. By the quadratic formula, this equation has two complex roots: $-2 \pm 3i$. Now, observe that $e^{(-2+3i)t} = e^{-2t} \cdot e^{3it} = e^{-2t}\cos(3t) + ie^{-2t}\sin(3t)$. By considering the real and imaginary parts by themselves, we obtain two real solutions to the differential equation.

$$\begin{aligned} x(t) &= e^{-2t}\cos(3t) \\ x(t) &= e^{-2t}\sin(3t) \end{aligned}$$

Therefore, the general solutions is obtained from these two particular solutions.

$$x(t) = C_1 e^{-2t}\cos(3t) + C_2 e^{-2t}\sin(3t)$$

This example illustrates the picture that we see in general: solutions to second-order homogeneous equations whose characteristic equation has two complex solutions are a sort of hybrid between exponential and sinusoidal functions. The exponential function determines a rate of decay (or explosion), while the sine and cosine factors determine an oscillation.

[1]Leonhard Euler (1707-1783) was a prolific Swiss mathematician. He was one of the first people to realize the importance of the notion of functions within mathematics.

6 One real root

The final case is the situation where the characteristic equation has only a single (real) root. The simplest example of this is equation (I), where that characteristic equation is $r^2 = 0$, which has only one root $r = 0$. This root does, in fact, give a solution $x(t) = e^{0t}$, also known as $x(t) = 1$. The other solution is, in fact, $x(t) = t$.

It turns out that the behavior of equation (I) really encapsulates what must always happen when the characteristic equation has only one root. This one root will give an exponential solution $x(t) = e^{rt}$, and the other solution will be te^{rt}.

This fact can be seen by doing a clever substitution. Rather than introduce complicated notation to prove this fact in general, I will illustrate the argument with an example. The argument will look exactly the same in all such cases.

Example 6.1. Consider the differential equation $x'' - 2x' + x = 0$.

The characteristic equation is $r^2 - 2r + 1 = 0$, which factors as $(r-1)^2 = 0$. So there is only one real root: $r = 1$. So we obtain only one solution this way: $x(t) = e^t$. In fact, another solution is $x(t) = te^t$.

To see why this is the case, do the following substation: $x(t) = e^t u(t)$. The idea is that we know that $u(x) = 1$ give one solution to the differential equation, and we are seeking another. Notice that $x'(t) = e^t u(t) + e^t u'(t)$, and $x''(t) = e^t u(t) + 2e^t u'(t) + e^t u''(t)$. Therefore, $x''(t) - 2x'(t) + x(t) = e^t u''(t)$. It follows that $x''(t) - 2x'(t) + x(t) = 0$ if and only if $e^t u''(t) = 0$, which is true if and only if $u''(t) = 0$. Of course, the only possibility here is that $u(t)$ is a linear function. In particular, $u(t) = t$ is another solution. From this, it follows that $x(t) = e^t u(t) = te^t$ is another solutions.

Since e^t, te^t are two independent solutions, combining them gives the general solution.

$$x(t) = C_1 e^t + C_2 te^t$$

Note that when writing up problems on your homework, you do not need to write this substation argument each time. In this case, it suffices to say that $r = 1$ is the only root of the characteristic equation, so $x(t) = e^t$ and $x(t) = te^t$ are both solutions.

7 Summary

The results of this lecture can be summarized as follows.

1. Given a second-order homogeneous equation $x'' + bx' + cx = 0$, the *characteristic equation* is the equation $r^2 + br + c = 0$.

2. Real solutions of the characteristic equation correspond to solutions of the differential equation.

3. If the characteristic equation has two real solutions r_1 and r_2, then the general solution of the differential equation is $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

4. If the characteristic equation has two complex solutions $\alpha \pm i\beta$, then the general solution of the differential equation is $x(t) = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$.

5. If the characteristic equation has only one real solution r, then the general solution of the differential equation is $x(t) = C_1 e^{rt} + C_2 t \cdot e^{rt}$.

8 Commentary

As we have seen, there are three cases that arise from second-order homogeneous equations.

As a first comment: you might worry that this situation will become only more complicated when considering higher-order equations (for example, $x''' + 7x'' - 2x' + 5x = 0$). Although we shall not study such equations in this course, the remarkable fact is the *the situation does not become any more complicated.* Again, there is a characteristic equation. Again, its real solutions give exponential solutions to the differential equation, while its complex solutions gives solutions involving oscillation. No more families of solutions are needed; all of the complexity is already present in the second-order case. Indeed, this is one of the main reasons that we have studied these equations in the first place: they exemplify all of the behavior that becomes important in studying linear differential equations (and linear systems of equations). The fact that the complexity ends at two is essentially what is called the fundamental theorem of algebra.

As a second comment, it is worth revisiting the spring interpretation of second-order homogenous equations, in light of the analysis performed here. In particular, recall that if $x'' + bx' + cx = 0$ is a differential equation, and b and c are positive, then this equation models the motion of a *damped spring*, where the constant c measures the elasticity of the spring (it is called the spring constant), and the constant b measures how much energy is dissipated to friction as the spring moves (it is called the damping constant).

How do different values of b and c lead to different sorts of solutions? Recall that by the quadratic formula, the roots of the characteristic equation are $r = \frac{-b\pm\sqrt{b^2-4c}}{2}$. Whether this are real or complex depends on the *discriminant* $b^2 - 4c$. Notice that if $b = 0$ (no damping), this is negative, and the characteristic equation has two imaginary roots. The solution is simply a sine wave. On the other hand, for b very large, $b^2 - 4c$ becomes positive, and the characteristic equation has two real roots. In this case, there are two independent exponential solutions; there is no oscillation. This is the case of an over damped spring; there is so much friction that the spring never moved past equilibrium to oscillate. All of this is in line with physical intuition.

My final comment is that there is one central lesson to these past two lectures: that the exponential function is a prototype for the functions that arise in simple (especially linear) differential equations. To understand such differential equations entirely revolves around understanding the exponential function, in both its real and complex guises.

9 Appendix: Euler's formula

There are several ways to conceive of Euler's remarkable formula.

$$e^{it} = \cos t + i \sin t$$

Perhaps the easiest route to explain (though in my opinion, ultimately the least revealing route) is via Taylor series.

The most reasonably definition of e^t is this: it is the unique function that is equal to its own derivative and is equal to 1 at $t = 0$. This forces the following Taylor expansion onto us.

$$e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \cdots$$

This Taylor series is so essential to the nature of e^t that is is unimaginable to part with it in any situation. In particular, it should also be true for imaginary inputs. Lets see what results from this. Notice that $i^2 = -1$, so $i^3 = -i$ and $i^4 = 1$. Therefore:

$$\begin{aligned} e^{it} &= 1 + it + \frac{1}{2}(it)^2 + \frac{1}{3!}(it)^3 + \frac{1}{4!}(it)^4 + \frac{1}{5!}(it)^5 + \cdots \\ &= 1 + it - \frac{1}{2}t^2 - i\frac{1}{3!}t^2 + \frac{1}{4!}t^4 + i\frac{1}{5!}t^5 + \cdots \\ &= (1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 - \cdots) + i(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots) \end{aligned}$$

Of course, these real and imaginary parts are easy to recognize as the Taylor series for the usual trigonometric functions. Euler's formula results.

A second explanation of Euler's formula can begin with the knowledge that $C_1 \sin t + C_2 \cos t$ is the general solution to $x''(t) + x(t) = 0$. Now, whatever e^{it} is, it must certainly have derivative ie^{it} and thus second derivative $-e^{it}$. So $x(t) = e^{it}$ should be a solution to $x''(t) + x(t) = 0$. Therefore $e^{it} = C_1 \sin t + C_2 \cos t$ for some values of C_1 and C_2. How can we determine C_1 and C_2? We simple supply initial conditions. Of course $e^{it} = 1$ at $t = 0$, and its derivative ie^{it} is equal to i at $t = 0$. Solving this initial value problem gives $C_1 = i$ and $C_2 = 1$. This again gives Euler's formula.

A third explanation is more geometric in nature. While it is, in my opinion, the clearest of the three explanations, it requires a degree of mathematical experience to see why. The basic observation, as in the previous explanation, is that the function $x(t) = e^{it}$ must satisfy $x'(t) = ie^{it}$. What does this mean? The function $x'(t)$ describes the velocity of the point $x(t)$ as t increases. Since we are working with complex numbers, this point $x(t)$ is described by a point in the plane. Thus the velocity of this point is given by i times its position. Geometrically, multiplication by i corresponds simply to a 90° counter-clockwise rotation. So the point $x(t)$, as t increases, always has velocity perpendicular to its position vector (measured from the origin). But there is a physical interpretation of this: the point $x(t)$ is moving exactly like an orbiting planet would. So as t increases, e^{it} will move in a circle around the origin. Its speed is proportional to the radius of the circle. Since $e^0 = 1$, this radius is 1. To summarize: the function e^{it} should describe counterclockwise motion at speed 1 along a circle of radius 1. This is essentially the definition of the trigonometric functions; so $e^{it} = \cos t + i \sin t$.