Lecture 29: Series solutions

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1 Introduction

Recall the following analogy: Taylor series are to functions what decimal expansions are to numbers. Just as we are usually content with the first several decimal places of a number arising from solving some equation, when solving differential equations it is sometimes sufficient to find the first several terms of the Taylor series, and using it as an approximation of a solution to the differential equation.

This lecture discusses how it is sometimes possible to extract a Taylor approximation of a function from a differential equation. This is sometimes useful from a computational standpoint (especially in situations where it would be difficult or impossible to solve the differential equation explicitly), and it also serves as an illustration of how power series and Taylor series occur elsewhere in mathematics.

This lecture will also introduce the study of second-order differential equations, that is, differential equations involving second derivatives as well as first derivatives. These will be studied in some depth in the next two lectures.

The reading for today is the handout "Series solutions to differential equations." The homework is problem set 28 (which includes weekly problems 26 and 27, although the problem sheet mistakenly neglects to mention number 26) and a topic outline.

2 The basic technique

The goal for today is to take an initial value problem (a differential equation and an initial condition) and to construct from it a Taylor approximation for the solution, centered around the initial condition.

Here is a strategy that will work in many cases.

1. Begin by assuming that the solution has power series representation $y(t) = a_0 + a_1 t + a_2 t^2 + \cdots$, where $a_0, a_1, \cdots$ are constants.

2. Plug this expression into the differential equation and initial condition to determine what they must be.

Notice that $y(0) = a_0$. Therefore we shall generally attempt to express all other coefficients in terms of a_0, since it is the initial condition for the differential equation.

Example 2.1. Consider the equation $y'(t) = y(t)$. As we know very well by now, the solutions to this equation are $y(t) = Ce^t$ for constants C. Here is how to deduce this by thinking about the Taylor series.

Suppose that $y(t)$ is a solution to this equation, and assume that $y(t)$ has Taylor expansion $y(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ around 0.

Now consider what the differential equation says, in terms of the Taylor series.

$$\begin{aligned} y'(t) &= y(t) \\ \left(a_0 + a_1 t + a_2 t^2 + \cdots\right)' &= a_0 + a_1 t + a_2 t^2 + \cdots \\ a_1 + 2a_2 t + 3a_3 t^3 + \cdots &= a_0 + a_1 t + a_2 t^2 + \cdots \end{aligned}$$

Since these two Taylor series represent the same function, all of their coefficients must be equal. Therefore:

$$\begin{aligned} a_1 &= a_0 \\ 2a_2 &= a_1 \\ 3a_3 &= a_2 \\ &\cdots \end{aligned}$$

Now observe that this means that every single coefficient is determined only by the first coefficient. Indeed, $a_1 = a_0$, then $a_2 = \frac{1}{2}a_1 = \frac{1}{2}a_0$, then $a_3 = \frac{1}{3}a_2 = \frac{1}{3}\frac{1}{2}a_0$, and so on. It is not hard to see the general pattern.

$$a_n = \frac{1}{n!}a_0$$

Therefore $y(t)$ has a Taylor series expansion given by the following.

$$\begin{aligned} &a_0 + a_0 t + \frac{a_0}{2}t^2 + \frac{a_0}{3!}t^3 + \cdots \\ &= a_0\left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \cdots\right) \end{aligned}$$

Indeed, this shows, as we have seen by other means, that all solutions of this differential equation have the form $y(t) = a_0 e^t$, for a constant a_0.

Example 2.2. Now consider the equation $y' = ky$, where k is a constant. We can apply the same reasoning to this equation. Suppose that $y(t) = a_0 + a_1 t + a_2 t^2 + \cdots$, and plug this into the differential equation.

$$\begin{aligned} y'(t) &= ky(t) \\ \left(a_0 + a_1 t + a_2 t^2 + \cdots\right)' &= k\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) \\ a_1 + 2a_2 t + 3a_3 t^3 + \cdots &= ka_0 + ka_1 t + ka_2 t^2 + \cdots \end{aligned}$$

Now, equate the coefficients one by one.

$$\begin{aligned} a_1 &= ka_0 \\ 2a_2 &= ka_1 \\ 3a_3 &= ka_2 \\ &\cdots \end{aligned}$$

Now, if we fix the first coefficient, these conditions determine all of the other coefficients.

$$\begin{aligned} a_1 &= ka_0 \\ a_2 &= \frac{k}{2}a_1 = \frac{k^2}{2}a_0 \\ a_3 &= \frac{k}{3}a_2 = \frac{k^3}{3 \cdot 2}a_0 \\ &\cdots \\ a_n &= \frac{k^n}{n!}a_0 \end{aligned}$$

Putting this coefficients back into the series, we obtain the following form for the solution as a power series.

$$a_0 \left(1 + kt + \frac{k^2}{2}t^2 + \frac{k^3}{3!}t^2 + \cdots\right)$$

You should check that this is the same thing as the Taylor series for $a_0 e^{kt}$.

Example 2.3. Consider the differential equation $y'(t) = 2t \cdot y(t)$. This equation is separable, so we could solve it using methods from the previous class. But we can also get a series solution for this equation fairly easily.

Begin as usual by assuming that $y(t) = a_0 + a_1 t + a_2 t^2 + \cdots$. Then we can write each side of this equation as a power series.

$$\begin{aligned} y'(t) &= a_1 + 2a_2 t + 3a_3 t^2 + \cdots \\ 2t \cdot y(t) &= 0 + 2a_0 t + 2a_1 t^2 + 2a_2 t^3 + \cdots \end{aligned}$$

Now, equate the terms of these two expressions, one by one.

$$\begin{aligned} a_1 &= 0 \\ 2a_2 &= 2a_0 \\ 3a_3 &= 2a_1 \\ 4a_4 &= 2a_2 \\ &\cdots \end{aligned}$$

Now, as usual let us attempt to use these equations to write all other coefficients in terms of the initial term a_0. This is slightly trickier in this case, but still straightforward.

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_0 \\ a_3 = \frac{2}{3}a_1 &= 0 \\ a_4 = \frac{2}{4}a_2 &= \frac{1}{2}a_0 \\ a_5 = \frac{2}{5}a_1 &= 0 \\ a_6 = \frac{2}{6}a_4 &= \frac{1}{3}\frac{1}{2}a_0 \\ a_7 = \frac{2}{7}a_5 &= 0 \\ a_8 = \frac{2}{8}a_6 &= \frac{1}{4}\frac{1}{3}\frac{1}{2}a_0 \end{aligned}$$

The pattern should be becoming clear. The even-degree coefficients are $a_{2n} = \frac{1}{n!}a_0$, while the odd-degree coefficients are 0. Hence here is the Taylor series.

$$a_0 + a_0 t^2 + \frac{1}{2}a_0 t^4 + \frac{1}{3!}t^6 + \cdots$$

Notice that this can also be written as follows.

$$a_0\left(1+t^2+\frac{1}{2}(t^2)^2+\frac{1}{3!}(t^2)^3+\cdots\right)$$

You should recognize this as the Taylor series for $a_0e^{t^2}$. Indeed, you would also discover this solution by separating the equation.

3 Non-separable examples

The examples so far have all been of a sort that we already knew how to solve, namely separable equations. here are some examples that cannot be solved by separation of variables, but are easy to solve by considering the series.

Example 3.1. Consider the equation $y' = x + y$, where y is a function of x.

Suppose that $y(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then we can express the two sides of this equation as follows.

$$\begin{aligned} y'(x) &= a_1 + 2a_2x + 3a_3x^3 + \cdots \\ x + y(x) &= a_0 + (a_1+1)x + a_2x^2 + a_3x^3 + \cdots \end{aligned}$$

Now equating these terms one by one results in the following.

$$\begin{aligned} a_1 &= a_0 \\ a_2 = \frac{a_1+1}{2} &= \frac{1}{2}(a_0+1) \\ a_3 = \frac{1}{3}a_2 &= \frac{1}{3!}(a_0+1) \\ a_4 = \frac{1}{4}a_3 &= \frac{1}{4!}(a_0+1) \end{aligned}$$

Therefore the Taylor series looks as follows.

$$a_0 + a_0x + \frac{a_0+1}{2}x^2 + \frac{a_0+1}{3!}x^3 + \frac{a_0+1}{4!}x^4 + \cdots$$

You might recognize this as a power series representation for $(a_0+1)e^x - x - 1$, which is indeed one way to write the general solution to this differential equation.

Example 3.2. Consider the differential equation $y' = 2xy + e^x$. This one is a bit more complicated, but the technique is the same. To save effort, let us only compute a third-order approximation to a solution of this function.

Assume that $y(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then as usual, the right hand side can be expressed as follows.

$$y'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

The right side can be written as a sum of two power series, which can then be combined.

$$\begin{aligned} 2xy(x) + e^x &= \left(2a_0x + 2a_1x^2 + 2a_2x^3 + \cdots\right) + \left(1 + x + \frac{1}{2}x^2 + \cdots\right) \\ &= 1 + (2a_0+1)x + (2a_1 + \frac{1}{2})x^2 + (2a_2 + \frac{1}{3!})x^3 + \cdots \end{aligned}$$

Now, equation the coefficients of these two expressions one by one to solve for the values $a_1, a_2, \cdots$.

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= \frac{1}{2}(2a_0 + 1) \\
&= a_0 + \frac{1}{2} \\
a_3 &= \frac{1}{3}(2a_1 + \frac{1}{2}) \\
&= \frac{2}{3} + \frac{1}{3!}
\end{aligned}
$$

It would be possible to keep going (and eventually to find a pattern and even a closed form solution), but for now let us be content to stop here with a third-order approximation.

$$y(x) \approx a_0 + x + (a_0 + \frac{1}{2})x^2 + (\frac{2}{3} + \frac{1}{3!})x^3$$

One of the strengths of the power series approach is that it can be used to get polynomial approximations to solutions, without necessarily totally solving the equation.

4 Second-order equations

Beginning in the next lecture, we will begin to study second-order differential equations. A second order differential equation is a differential equation which involved second, as well as first, derivatives. As a brief introduction, here is a first example of such an equation.

Example 4.1. Find all solutions to the differential equation $y'' = y$, where y is a function of t.

Let us use the same strategy as before: begin by assuming that $y(t) = a_0 + a_1 t + a_2 t^2 + \cdots$. Then $y''(t) = 2a_2 + 3 \cdot 2a_3 t + 4 \cdot 3a_4 t^2 + 5 \cdot 4a_5 t^3 + \cdots$.

As before, let us equation these two series, term by term.

$$
\begin{aligned}
2a_2 &= a_0 \\
3 \cdot 2a_3 &= a_1 \\
4 \cdot 3a_4 &= a_2 \\
&\cdots
\end{aligned}
$$

Now, we can solve for most of the terms of the series.

$$
\begin{aligned}
a_2 &= \frac{1}{2}a_0 \\
a_3 &= \frac{1}{3\cdots 2}a_1 \\
&= \frac{1}{3!}a_1 \\
a_4 &= \frac{1}{4\cdot 3}a_2 \\
&= \frac{1}{4!}a_0 \\
a_5 &= \frac{1}{5\cdot 4}a_3 \\
&= \frac{1}{5!}a_1 \\
a_6 &= \frac{1}{6\cdot 5}a_4 \\
&= \frac{1}{6!}a_0
\end{aligned}
$$

Now, there is a pattern beginning to emerge, but it slightly peculiar: each term is written either $a_n = \frac{1}{n!}a_0$ (when n is even) or $a_n = \frac{1}{n!}a_1$ (when n is odd). There is no way to solve for a_1 in terms of a_0; both variables are free. In fact, we can write the solution to this equation as follows.

$$y(x) = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots\right) + a_1\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)$$

Observation 4.2 (not for exam). The example above shows that all solutions to $y'' = y$ arise from combining two functions. These two functions actually have names: they are called the hyperbolic trigonometric functions, and are denoted as follows. They are named because of their formal similarity to the usual trigonometric functions: if these series are modified so that the terms alternate signs, one obtains the Taylor series for $\cos x$ and $\sin x$.

$$
\begin{aligned}
\cosh x &= 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots \\
\sinh x &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots
\end{aligned}
$$

You may also verify, if you wish, that these functions can be defined without series as follows: $\cosh x = \frac{1}{2}(e^x + e^{-x})$, and $\sinh x = \frac{1}{2}(e^x - e^{-x})$. Notice that $\frac{d}{dx}\cosh x = \sinh x$ and $\frac{d}{dx}\sinh x = \cosh x$, which is another formal similarity with $\sin x$ and $\cos x$.

We saw in the previous example that it takes two parameters, rather than one, to specify a solution to a second-order equation (in this case, a_0 and a_1). Indeed, given a differential equation like $y''(x) = y(x)$, I must specify not just $y(0)$, but also $y'(0)$, in order to uniquely determine a solution of the differential equation.

There is a physical interpretation to this: a second-order differential equation can be thought of as determining the acceleration of a particle, which is to say the amount of force applied to it. In order to track the trajectory of a particle under the influence of various forces, I need to know not just where it starts, but how fast it is moving when it starts.

Definition 4.3. A second-order *initial value problem* consists of a second-order differential equation for a function $y(x)$, a chosen value $y(0)$, and a chosen value $y'(0)$.

As with first-order equations, there is a unique solution to a second-order initial value problem.

Example 4.4. Find the solution to the following initial value problem.

$$\begin{aligned} y''(x) &= y(x) \\ y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

As shown in the previous example, the solution function has the following Taylor series, for some constants a_0, a_1.

$$y(x) = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots\right) + a_1\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)$$

Now, observe that it follows from this that:

$$\begin{aligned} y(0) &= a_0 \\ y'(x) &= a_0\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right) + a_1\left(1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots\right) \\ y'(0) &= a_1 \end{aligned}$$

Therefore the unique solution with $y(0) = 1$ and $y'(0) = 2$ is the following.

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots\right) + 2\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)$$