Lecture 24: Series finale

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1 Introduction

This lecture will discuss one final convergence test and error bound for series, which concerns alternating series. A general summary of various concepts from the series unit follows.

The alternating series test gives a simple criterion for the convergence of a series whose terms alternate between positive and negative: if the terms decrease in magnetite and approach 0, then the series converges. This is most easily visualized by partial sum diagrams on the number line.

The next lecture will begin the unit on differential equations.

The reading for today is Gottlieb §30.4 up to page 956, and the handout titled "Alternating series" (under "reading for the course"). The homework is problem set 23 and a topic outline.

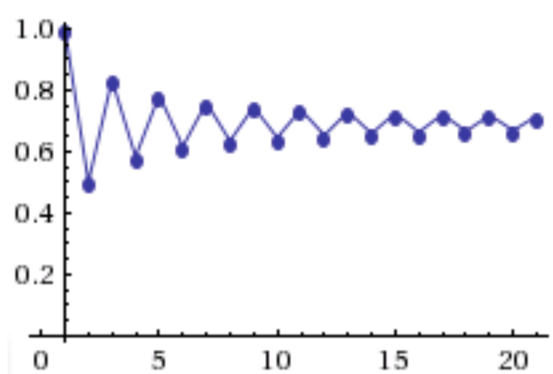
2 The alternating series test

The notion of an alternating series is quite simple. A series is called an *alternating series* if its terms alternative between being positive and negative. For example, the following are alternating series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
$$1 - 1 + 1 - 1 + \cdots$$
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

When a series is written in Σ notation, alternating series are most easily recognized by a factor of $(-1)^n$, as in the third example above. Alternating series frequently occur in the context of power series.

It turns out to often be much simpler to consider alternating series than series in general. For example, consider the alternating harmonic series. To help visualize the sums of this series, the plot below shows successive partial sums.



Two features are apparent: the partial sums are bouncing back and forth around the eventual sum (alternating between being above and below it), and each sum is either an upper bound or lower bound for all the sums that follow it.

What caused this to happen in the picture? The essential fact is that the magnitudes of the terms are *decreasing*. Reasoning in this way give the following result.

Theorem 2.1 (Alternating series test). *If the terms of an alternating series are decreasing in magnitude and approach* 0*, then the series converges. In symbols, a series* $\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \cdots$*, where each* a_n *is positive, will converge if* $a_0 > a_1 > a_2 > \cdots$ *and* $\lim_{n\to\infty} a_n = 0$*.*

In fact, a little bit more can be said about such alternating series: it is very easy to obtain error bounds for partial sums (i.e. bounds on the distance between a partial sum and the eventual sum).

Suppose that $a_0 - a_1 + a_2 - a_3 + \cdots$ is an alternating series with $a_0 > a_1 > a_2 > \cdots$. Denote by s_n the partial sum $s_n = a_0 - a_1 + a_2 - \cdots \pm a_n$. Then notice that if n is even, then s_n is an upper bound for all later partial sums, while if n is odd then s_n is a lower bound. In particular, the eventual limit (if it exists) must lie somewhere between s_n and s_{n+1}. This means, in particular, that the error of s_n (the difference between is and the total sum) is bounded by $|s_{n+1} - s_n| = a_{n+1}$. Thus we obtain the following.

Theorem 2.2 (Alternating series error estimate). *Suppose* $a_0 - a_1 + a_2 - a_3 + \cdots$ *is an alternating series with* $a_0 > a_1 > a_2 > \cdots$ *and such that* $\lim_{n\to\infty} a_n = 0$*. Let the sum of this series be* s*. Then the error of the partial sum* s_n *is at most* $|a_{n+1}|$*. In other words,*

$$|(a_0 - a_1 + a_2 - \cdots \pm a_n) - s| < a_{n+1}$$

In other words: the error of a partial sum of the series is bounded by the next term in the series.

Example 2.3. We have mentioned that the sum of the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is $\ln 2$ (this comes from the Taylor series of $\ln x$ centered at 1). How many terms must be added to guarantee an error of less than $1/100$ from the actual value of $\ln 2$?

Consider a partial sum $1 - \frac{1}{2} + \cdots \pm \frac{1}{n}$. By he alternating series error estimate, the difference between this value and $\ln 2$ is bounded in magnitude by the absolute value of the next term, i.e. $\frac{1}{n+1}$. If we want this error to be less than 0.01, we must ensure that $\frac{1}{n+1} \leq \frac{1}{100}$, i.e. $n \geq 99$.

So $1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{99}$ estimates $\ln 2$ to within an error of 0.01. Note that this is quite a bit of computation to have to do for this level of accuracy – in practice, more efficient methods should be used.

Note also that since the next terms to be added is $-\frac{1}{100}$, this partial sum is in fact an *upper bound* for the actual value $\ln 2$.

Example 2.4. Estimate $\frac{1}{e}$ to within an error of 0.01.

We can use the Maclaurin series for e^x to see that $\frac{1}{e} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \cdots$. This is an alternating series with terms decreasing in magnitude and approaching 0. The partial sum $1 - 1 + \frac{1}{2} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$ must have error of at most the size of the next term, $\frac{1}{(n+1)!}$. Therfore, we need n to be large enough that $\frac{1}{(n+1)!} < \frac{1}{100}$, i.e. $(n+1)! > 100$. Since $5! = 120 > 100$, $n = 4$ will suffice.

Therefore the sum $1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} = \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8}$ is an estimate of $\frac{1}{e}$, with error of at most $\frac{1}{120} < 0.01$. In fact, it is an overestimate since the last term included was positive.

Note, in fact, that this error bound is the same as what would be obtained from the Taylor remained theorem.

3 Glossary of series terms

The following glossary lists and gives brief descriptions of most of the important terms from the unit on series.

- *Absolute convergence* A series is absolutely convergent if the sum of the absolute values of the terms converges. Absolute convergence implies convergence.
- *Alternating series.* A series whose terms alternative between positive and negative.
- *Alternating series error estimate.* An error bound for the partial sums of an alternating series. If a series passes the alternating series test (see below), then the error of a partial sum (distance from it to the true sum) is less than the size of the next term.
- *Alternating series test.* A convergence test. If an alternating series has terms approaching zero and decreasing in magnitude, then it converges.
- *Asymptotics.* A variation on the comparison test, useful for studying sums of complicated expressions. Given two series that are asymptotic (denoted $a_n \sim b_n$), if the terms of b_n are positive, then one series converges if and only if the other does.
- *Comparison test.* A method to show convergence or divergence, applicable only to series with positive terms. If one series bounds another, than divergence of the smaller series means divergence of the larger, and convergence of the larger series means convergence of the smaller.
- *Conditional convergence.* A series converges conditionally if it converges, but it is not absolutely convergent (for example, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$).
- *Convergence.* A series converges if its partial sums approach a limit, which is called the sum.
- *Geometric series.* A simple type of series, often used for comparison. A series is geometric if all paris of consecutive terms have the same ratio. Such series converge if and only if the ratio r is less than 1 in magnitude, in which case the sum is equal to the first term multiplied by $\frac{1}{1-r}$.
- *Harmonic series.* A famous divergent series. It is $1 + \frac{1}{2} + \frac{1}{3} + \cdots$. It diverges to infinity, albeit very slowly (the nth partial sum is within 1 of $\ln n$).
- *Initial segments don't matter.* A mantra. While initial terms in a series may drastically affect its sum, they do not affect convergence: one may always throw out any number of initial terms before studying convergence.
- *Integral test.* A method to show convergence or divergence. If the terms of a series are given by values of a positive and decreasing function, then the series converges if and only if the improper integral of the function converges.
- *Interval of convergence.* The set of all values where a power series converges.
- *Maximal open interval of convergence.* The largest open interval on which a power series converges. It is determined entirely by the radius of convergence and the center, as $(c - R, c + R)$. It is preserved under differentiation and integration of the power series.
- *Nth term test.* A convergence criterion. If the terms of a series do not approach 0, then the series diverges. If they do approach 0, the series may diverge or may converge.
- *Partial sum.* The sum of the first n terms of a series. A series converges if and only if the partial sums approach a limit as $n \to \infty$.
- *Power series.* A series whose terms include a factor $(x-c)^n$ (where c is a constant, and x is a variable). Substituting a chosen value of x gives a series in the ordinary sense, which may converge or diverge depending on the value of x.
- *Power series representation.* A power series which converges in some interval around the center to a particular function $f(x)$. Such a representation is unique; it must be the Taylor series.

- *p-series* An example of uncreative naming. A p-series is a series of the form $\sum \frac{1}{n^p}$. Such a series converges if and only if $p > 1$.
- *Ratio test.* A convergence test, based on comparison to geometric series. If the limit of the ratios of the absolute values of consecutive terms approach a limit L, then $L < 1$ implies the series converges absolutely, $L > 1$ implies that the series diverges, and $L = 1$ is inconclusive.
- *Radius of convergence.* A way to describe the convergence of a power series. It is a threshold R, such that the power series converges absolutely for inputs less than this distance, and diverges for inputs that are greater. The value R is preserved under integration and differentiation of the series.
- *Ratio test.* A convergence test, based on comparison to geometric series. If the limit of the nth roots of the absolute values of terms approach a limit L, then $L < 1$ implies the series converges absolutely, $L > 1$ implies that the series diverges, and $L = 1$ is inconclusive.
- *Series.* An infinite sum.
- *Taylor series.* A type of power series, obtained by first choosing a function and a center, and selecting the terms to be $\frac{f^{(n)}(c)}{n!}(x-c)^n$. If the function has a power series representation, this is the only possible series that it could be.
- *Zeno's paradox.* An early and influential example of series being confusing.

  I probably have forgotten some important terms and examples; feel free to point out more things I should add.

4 A series flowchart

The following is a rough list of steps to follow to try to determine whether a given series converges. It is kind of like a choose your own adventure. Of course you

Suppose that the series is given as

$$a_1 + a_2 + a_3 + \cdots .$$

1. Do you recognize the series from somewhere (p-series, geometric, value of a Taylor series evaluated at some point)?
2. (N^{th} term test) Do the terms approach 0, i.e. is $\lim_{n\to\infty} a_n = 0$? If not, the series **diverges**. If so, continue.
3. (Alternating series) Is your series alternating? If so, see if the terms are decreasing in size. If so, the series **converges**. If not, continue.
4. (Absolute values) Replace all the terms of the series with their absolute values, go back to step 1 and see if this new series converges. If so, then the series **converges absolutely**. If not, then it may still converge conditionally, but unless the alternating series test worked, you may have to be clever to show this.
5. (Comparison) If the series has positive terms, see if you can simplify it asymptotically, compare it to another simpler series, or (in case the terms are decreasing) compare it to an improper integral.
   - Comparing to integrals, the integral converges if and only if the series converges.
   - In asymptotic comparison of positive series, one series converges if and only if the other converges.
   - When directly comparing (using $\leq$ or $\ll$), the smaller series will converge if the larger converges, and the larger will diverge if the smaller diverges.

6. (Ratio and root tests) See if you can calculate $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ (if both exist, then they are equal). If the limit is L, and $L < 1$, then the series **converges absolutely**. If $L > 1$, the series **diverges**. Otherwise, the test is inconclusive.

These methods summarize most of the main techniques that we have studied. If none of them work, you may have to get more creative.