Lecture 23: Power series II

Nathan Pflueger

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1 Introduction

In the previous lecture, we considered power series, i.e. series which varied in a specific way with a variable x, and the fact that such series may converge for some values of x and diverge for others. In this lecture, we shall consider what happens when we regard these series as functions of x and manipulate them using the main tools of calculus. Specifically, we are interested in the effect of differentiation and integration on power series.

The results are roughly what one might expect: power series can be differentiated or integrated one term at a time. One need only worry about convergence, but this turns out to be straightforward: although these operations may change the interval of convergence, the *radius* of convergence is unchanged. The only possible changes occur at the endpoints.

Again, you should remember that "power series" is nearly synonymous with "Taylor series" (the difference being that a power series is specified as series and may define a function, whereas a Taylor series begins with a function that determines a power series). The reason these two end up being nearly synonymous is that a power series representation of a given function is unique, as we will discuss.

The reading for today is Gottlieb §30.4, beginning at "Manipulating power series" on page 956, as well as the "Power series and functions" handout (under "reading for the course"). You should not spend too much time worrying about the last several pages of the reading from Gottlieb (on differential equations), since we will not cover it in much detail just yet. The homework is problem set 22 (which includes weekly problem 24) and a topic outline.

2 Power series representation of a function

Suppose that I write down a power series, and claim that it is the definition of a function.

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

It is clear what is meant by this: the value of $f(x) = e^x$ at a given value of x is given by substituting that value into the power series and computing the infinite sum. When a function is described in this way, it is called a *power series representation* of the function. For example, the series above is a power series representation of e^x with center 0[1].

I call it *a* representation, but in fact I could say that it is *the* representation with center 0. This is because if a function is represented by any power series with a given center, that series must be the Taylor series. This is discussed in section 5.

Note that such a definition only makes sense for value of x within the radius of convergence. In this example, the radius is ∞, so this representation works everywhere. This is not necessarily the case.

[1] In fact, this power series is often used as the *definition* of the exponential function, since it is somewhat more efficient than any other definition.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

This power series gives a power representation of the function $\frac{1}{1-x}$, but it is only valid for $|x| < 1$. So power series representation does not necessarily define a function everywhere – only within the series's interval of convergence.

3 Differentiating power series

Given a power series representation of a function, the derivative of the function has a power series representation, which can be obtained by differentiating the terms of the power series individually. For example, consider the following.

$$\begin{aligned}
\sin x &= x - \frac{1}{3!}^{3} + \frac{1}{5!}x^5 - \cdots \\
\frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(x - \frac{1}{3!}^{3} + \frac{1}{5!}x^5 - \cdots\right) \\
&= \frac{d}{dx}(x) + \frac{d}{dx}\left(-\frac{1}{3!}x^3\right) + \frac{d}{dx}\left(\frac{1}{5!}x^5\right) + \cdots \\
&= 1 - \frac{3}{3!}x^2 + \frac{5}{5!}x^4 + \cdots \\
&= 1 - \frac{1}{2!}x + \frac{1}{4!}x^4 + \cdots \\
&= \cos x
\end{aligned}$$

The final line follows from knowing that $\cos x$ is represented by its Taylor series for all values of x.

In fact, this technique will always work. Furthermore, differentiating term by term does not change the radius of convergence. We summarize this in a theorem.

Theorem 3.1. *Suppose that a function $f(x)$ is represented by a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, which has radius of convergence $R > 0$. Then $f'(x)$ is represented by the power series obtained by differentiating each term individually, i.e. $\sum_{n=1}^{\infty} a_n \cdot n \cdot (x-c)^{n-1}$. The radius of convergence of this series is also R.*

Why does the radius of convergence stay the same? The reason is essentially the ratio test: multiplying the n^{th} coefficient by n will not affect the limit of the ratios of consecutive terms. Notice, however, that this does *not* mean that the interval of convergence is the same; only the radius.

Example 3.2. Consider the following two power series representations, which we have considered before.

$$\begin{aligned}
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots
\end{aligned}$$

The second is obtained by differentiating the first term by term. Both series have the same radius of convergence (namely 1). However, the interval of convergence of the first series is $(-1, 1]$, while the interval of convergence of the second series is $(-1, 1)$.

Sometimes, differentiating known power series is an easy way to find new power series representations.

Example 3.3. Find a power series expansion for $\frac{1}{(1+x)^2}$.
Observe that $\frac{d}{dx}\frac{1}{1+x} = -\frac{1}{(1+x)^2}$. Now reason as follows.

$$\begin{aligned}
\frac{1}{(1+x)^2} &= -\frac{d}{dx}\frac{1}{1+x} \\
&= -\frac{d}{dx}\left(1 - x + x^2 - x^3 + \cdots\right) \\
&= 0 + 1 - 2x + 3x^2 - \cdots \\
&= 1 - 2x + 3x^2 - \cdots
\end{aligned}$$

This series must also have radius of convergence 1, since the original series did. In fact, it has the same interval of convergence as well, namely $(-1, 1)$.

4 Integrating power series

Just as power series can be differentiated term by term, they can also be integrated term by term, in exactly the same way. For example, this is probably the easiest way to get a power series representation for the logarithm function.

$$\begin{aligned}
\ln(1+x) + C &= \int \frac{1}{1+x} dx \\
&= \int \left(1 - x + x^2 - x^3 + x^4 - \cdots\right) dx \\
&= \int 1dx - \int xdx + \int x^2dx - \int x^3dx + \int x^4dx - \cdots \\
&= C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots
\end{aligned}$$

Given that $\ln(1+0) = \ln 1 = 0$, we can take $C = 0$ and obtain the (previously known) fact:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Of course, this expression is only true within the interval of convergence. The radius of conference of the original series $1 - x + x^2 - x^3 + \cdots$ was 1, hence this is also the radius of convergence of the new series $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$. It is worth pointing out, however, that the *interval* of convergence is *not* the same. In fact, the interval of convergence of the original series was $(-1, 1)$, while the interval for the new series is $(-1, 1]$.

The main result here is very much analogous to the main result in the previous section.

Theorem 4.1. *Suppose that a function $f(x)$ is represented by a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, which has radius of convergence $R > 0$. Then $\int f(x)$ is represented by the power series obtained by integrating each term individually, i.e. $\sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot (x-c)^{n+1} + C$. The radius of convergence of this series is also R.*

This technique is sometimes useful for finding power series expansions of new functions from power series expansions of known functions, without troubling with troublesome derivative calculations. The following example was done in an earlier lecture, but we now have the necessary facts to justify it.

Example 4.2. Find a power series representation for $\tan^{-1} x$.
Use the fact that $\tan^{-1} x + C = \int \frac{1}{1+x^2}$, and the usual power series for $\frac{1}{1-x}$. We know that $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ is a power series representation, with radius of convergence $(-1, 1)$. Substituting $-x^2$ for x gives the following.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

This converges for $|-x^2| < 1$, i.e. for $|x| < 1$. So the radius of convergence is still 1. Now integrate this term by term.

$$\begin{aligned} \int \frac{1}{1+x^2} dx &= \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) \\ \tan^{-1} x + C &= C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \end{aligned}$$

Since $\tan^{-1} 0 = 0$, we can take $C = 0$ on the right, and thus obtain

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

The radius of convergence is still 1, so this certainly converges on $(-1, 1)$. In fact, the actual interval of convergence is $(-1, 1]$, but I shall not prove that at this time.

5 Uniqueness of power series representation

Power series representations of functions are essentially unique. More precisely, the following theorem holds.

Theorem 5.1. *Suppose that a function $f(x)$ has a power series expansion $a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$ around a center c, with radius of convergence R. Then if $R > 0$, then this is the unique power series expansion of $f(x)$ with center c and nonzero radius of convergence.*

The reason for this rather irritating qualification about positive radius of convergence is that a power with radius of convergence 0 is rather uninteresting – no terms are ever relevant except the first, since they are all 0 at the only point where the series converges.

Note the following corollary.

Corollary 5.2. *If the Taylor series of a function $f(x)$ around center c has nonzero radius of convergence, then it is the unique such power series representation of $f(x)$ around c.*

This corollary follows because taking derivatives and evaluating at 0 uniquely determines all of the coefficients of the power series.

For most well-behaved functions, the Taylor series expansion does indeed have nonzero radius of convergence. Thus for these functions, the notions of power series representation and Taylor series are identical.