# Lecture 19: Comparing to integrals; $p$-series

Nathan Pflueger

21 October 2011

## 1 Introduction

It has come up a number of times that series are analogous, in terms of technique and structure, to improper integrals. In this lecture, we state a theorem which makes the analogy explicit in many cases. This theorem can be used, in fact, to establish the convergence or diverges of most of the series that we have studied so far.

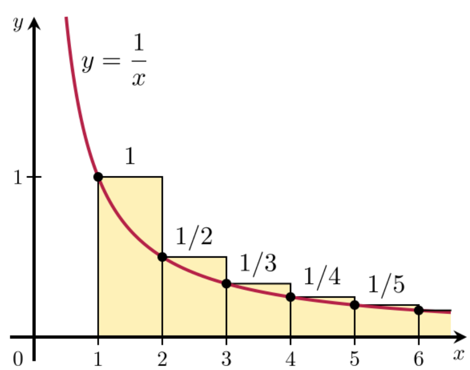
As an example, we will apply this criterion to a specific kind of series called $p$-series. These are series of the form $\sum \frac{1}{n^p}$, where $p$ is some constant. These series arise in somewhat surprising contexts, but are often difficult to evaluate explicitly. In an appendix, I will sketch an argument for summing such a series in the case $p = 2$.

The reading for today is Gottlieb §30.5, pages 968 to the top of 969 (although I would suggest that you begin reading at page 965, "the integral test") and the solutions to Janet's handout "$p$-series, comparing to improper integrals." The homework is problem set 18 and a topic outline. You should also begin working on weekly problems 21 and 22.

## 2 The integral test

The integral test states that, with relatively mild assumptions, many series with positive terms will converge if and only if a similar improper integral converges.

As an example, consider the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$. This can also be written as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. This series diverges, as does the integral $\displaystyle\int_1^{\infty} \frac{1}{x} dx$. Both of these facts follow from the other in a fairly natural way. Consider the following image (which I have shamelessly poached from the wikipedia article on the integral test[1].




[1]Incidentally, if you are not in the habit of looking up mathematical notions on wikipedia, you should know that it is a great source. Many of the math articles are quite well written, and give a lot of context.

This image suggests how this argument should go. On the one hand, the area under the curve $y = \frac{1}{x}$, up to $x = n$, is clearly less than the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. So given that this integral goes to infinity as $n$ grows, so too must the partial sums of the harmonic series.

On the other hand, we also could have run the argument in the opposite direction: suppose that we know that the harmonic series diverges, and we wish to show that the integral of $\frac{1}{x}$ also diverges. Then by using right-hand sums, rather than left-hand sums, you can observe that the integral $\int_1^n \frac{1}{x}dx \leq \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}$. Since these partial sums go to infinity as $n$ grows, so too must these integrals. So we could conclude that $\int_1^\infty \frac{1}{x}dx$ diverges.

If you think through precisely what makes these two arguments work, it is that $\frac{1}{x}$ is a *decreasing* function. In fact, our version of the integral test is the following.

**Theorem 2.1** (The integral test)**.** *Suppose that $f(x)$ is a function that is positive and decreasing for sufficiently large $x$*[2]*. Then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^\infty f(x)dx$ converges.*

Note that it is totally irrelevant that these integrals start at 0. You could replace $\int_1^\infty$ by $\int_0^\infty$ or $\int_{51}^\infty$, and the theorem is still true. This is part of the general principle: initial segments don't matter. So in the examples below, I will freely begin my integrals wherever is most convenient.

In fact, the integral test is really just a special case of the direct comparison test. I shall give a brief argument. Observe the following two inequalities hold if $f(x)$ is positive and decreasing.

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n) \tag{1}$$

Next, notice that if one defines $a_n = \int_n^{n+1} f(x)dx$, then $\sum_{n=1}^N a_n = \int_1^{N+1} f(x)dx$. Applying the comparison test to the series $\sum_{n=1}^\infty a_n$, it follows that if $\sum_{n=1}^\infty f(n)$ converges, then so does $\int_1^\infty f(x)dx$. On the other hand, using the left side of this inequality, the comparison test shows that if $\int_1^\infty f(x)dx$ converges, then so does $\sum_{n=1}^\infty f(n+1)$. Of course, this is the same as $\sum_{n=2}^\infty f(n)$, which converges if and only if $\sum_{n=1}^\infty f(n)$ converges. Therefore, the sum converges if and only if the integral converges.

# 3 First examples

The following examples are familiar already.

*Example* 3.1. Consider the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$. Consider the function $f(x) = \frac{1}{x}$. This function is positive and decreasing, so the integral test shows that this series converges if and only if $\int_1^\infty \frac{1}{x}dx$ converges. But $\int_1^b \frac{1}{x}dx = \ln|b|$, which goes to infinity (albeit slowly) as $b$ goes to infinity. So this integral diverges, and therefore the harmonic series diverges also.

*Example* 3.2. Consider the geometric series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$. The function $\left(\frac{1}{2}\right)^x$ is decreasing and positive for $x \geq 0$, so we may apply the integral test. Now, $\displaystyle\int_0^\infty \left(\frac{1}{2}\right)^x dx = \lim_{b\to\infty} \int_0^b \left(\frac{1}{2}\right)^x dx = \lim_{b\to\infty} \frac{1}{\ln\frac{1}{2}}\left(\frac{1}{2^b} - 1\right)$. As $b \to \infty$, this limit converges (since $\frac{1}{2^b} \to 0$), so the integral converges. In Therefore the sum also converges. Of course, we knew these by much more elementary means as well.

*Example* 3.3. In fact, since the integral of the function $f(x) = r^x$, where $r$ is a positive constant, is $\frac{1}{\ln r}r^x + C$, the same reasoning as above can be used to show that the sum $\sum_{n=0}^\infty r^n$ converges if and only if $r < 1$. This argument does not really work for negative values of $r$, since in these cases the function $f(x) = r^x$ does not make sense.

[2]The version in the book assumes that the function is decreasing and positive for all $x \geq 1$, but since initial segments don't matter, it actually suffices for this to hold for sufficiently large $x$.

## 4 $p$-series

A $p$-series is simply a series that looks like $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ (or, in other notation, $\sum_{n=1}^{\infty} n^{-p}$), where $p$ is some constant. For example, taking $p = 1$ gives the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, which we know diverges.

I am not sure why the letter $p$ is usually used, nor why, historically, no one thought of a more interesting same for these series. They happen to come up in interesting ways in higher mathematics, although for our purposes they are mainly only interesting as examples of the general principles of infinite series.

In fact, it is very easy to tell whether a $p$-series converges. The function $f(x) = x^{-p}$ is laughable easy to integrate: its integral is $\frac{1}{1-p}x^{1-p} + C$ (unless $p = 1$, in which case the integral is $\ln|x| + C$), and therefore unless $p = 1$, $\int_0^{\infty} x^{-p} = \lim_{b\to\infty} \frac{1}{1-p} b^{1-p}$, which exists if and only if $1 - p \leq 0$, i.e. $p \geq 1$. Of course, we excluded the case $p = 1$, when the harmonic series diverges. So the conclusion is the following.

**Theorem 4.1.** *The "p-series" $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.*

Evaluating the sums of $p$-series, on the other hand, is often not so easy (or even possible). Often the best way to sum them is, in fact, to simply add up the sum until the desired accuracy is reached. In the appendix that follows, though, I will give an amusing argument, touching on Taylor series (which we have been neglecting too much recently), to evaluate one particular $p$-series.

## 5 Appendix: the sum $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

By the integral test, the sum $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ converges. In fact, it converges to a rather surprising value. Here is a clever argument, which requires quite a bit more work to make watertight (in particular, it is necessary to work with calculus over complex numbers), but which gives the correct answer.

Recall that the function $f(x) = \sin x$ is equal to its Taylor series.

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

Now let us do something totally ridiculous: let us take treat the Taylor series as an infinite polynomial, and *factor* it. In order to do this, we need to know where $\sin x = 0$. Fortunately, this is easy: $\sin x = 0$ precisely whenever $x$ is an integer multiple of $\pi$. So we can try to "factor" $\sin x$ in something like the following way.

$$x(x - \pi)(x + \pi)(x - 2\pi)(x + 2\pi)(x - 3\pi)(x + 3\pi)\cdots$$

This way, however, is not quite right, because if we try to multiply this out, it will blow up to infinity. A better option is the following, which happens to be a true statement.

$$\sin x = x\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots$$

Now, this factorization gives the right zeros. The only question is whether it has the right constant factor. In fact, multiplying out the expression will give $x \cdot 1 \cdot 1 \cdot 1 \cdots = x$ as the first term, which is correct.

Now, we can rewrite this expression by grouping adjacent terms.

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots$$

Now, what is the $x^3$ term of this product? If you think about it, the only way that an $x^3$ term can appear is by multiplying together one of the $\frac{x^2}{n^2\pi^2}$ terms with all other 1 terms. The conclusion is that the first two terms of this product are:

$$\sin x = x - \left( \frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \cdots \right) x^3 + \cdots$$

On the other hand, from the Taylor series around $x = 0$, we know that the series begins $x - \frac{1}{6}x^3$. Therefore, it follows from this that:

$$\frac{1}{6} = \frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \cdots$$

Multiplying both sides by $\pi$ we obtain the following.

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Despite the many loose steps employed here, this is the correct sum.