Math 1B, lecture 15: Taylor Series

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1 Introduction

Taylor's theorem shows, in many cases, that the error associated with a Taylor approximation will eventually approach 0 as the degree of the approximation gets larger and larger. In other words, as more and more terms are added to the summation, the result becomes closer and closer to the value of the function.

Under the assumption that the error approaches 0 as n grows, Taylor series sidestep the need for error bounds by simply including infinitely many terms in the sum. Intuitively, the Taylor series of a function is the Taylor polynomial of infinite degree.

Formally, a Taylor series is a notational device which specifies each derivative of a function at a single point. In many cases, the infinite sum has a well-defined value, equal to the function itself. Thus Taylor series demonstrate, for well-behaved functions at least[1], that the data of all of the derivatives of a function at one point completely determine the values of the function at nearby points.

In this class, we will define Taylor series and give some examples. Questions of convergence are not as important for the time being. In subsequent classes we will begin a systematic study of convergence of infinite series, with Taylor series being the main example of interest.

The reading for today is Gottlieb §18.4 (on summation notation) and §30.3 up to page 944 (on Taylor Series). The homework is problem set 14. You should begin working on weekly problem 15.

2 Definition of a Taylor Series

Recall once more that the nth order Taylor approximation of a function $f(x)$, with center c, is a polynomial function defined as follows.

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

In the special case $c = 0$, this takes the particularly simple form below.

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

These approximations are chosen so that they:

- Are polynomials of degree n, and
- Match the function $f(x)$ and its first n derivatives at $x = c$.

A *Taylor series* can be thought of as a Taylor polynomial of infinite degree. The definition is as follows.

[1]Almost all functions we consider in this course are "well-behaved" in this sense. In higher mathematics, such functions are called *analytic functions.*

Definition 2.1. The *Taylor series* of a function $f(x)$ around a center c is the infinite sum:

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots.$$

Alternatively, in summation notation:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Note. In the notation above, I am using the following conventions.

$$\begin{aligned} f^{(0)}(x) &= f(x) \\ 0! &= 1 \end{aligned}$$

This is done simply so that the $k=0$ term of the sum will make sense. Intuitively, the 0^{th} derivative of a function is the function itself (it has been differentiated 0 times), and the number 0! should be 1! divided by 1 (just like $4! = (5!)/5$).

Notice that at the moment, the Taylor series is merely an infinite sum, which may or may not converge to a meaningful value. Much of the work of the rest of this unit will be studying when it does converge to a meaningful value.

Notice that *the information of a Taylor series is precisely equivalent to the value of a function and all of its derivatives at a single point.* Therefore whenever we study Taylor series, we are essentially studying the influence of the derivatives of a function at one point on the function at other points.

The most common center to use for Taylor series is, naturally, $c=0$. These are common enough that they commonly have their own name, which you should be aware of.

Definition 2.2. The *Maclaurin series* of a function $f(x)$ is the Taylor series at center $c=0$.

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

3 Examples

Consider four standard functions: $e^x, \sin x, \cos x$, and $\log(1+x)$. Let us compute the Maclaurin series (i.e. Taylor series around 0) for each of these functions. First, notice that all four functions have derivatives following a fairly simple pattern.

$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(5)}(x)$	$\cdots$
e^x	e^x	e^x	e^x	e^x	e^x	$\cdots$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$	$\cdots$
$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$	$-\sin x$	$\cdots$
$\log(1+x)$	$(1+x)^{-1}$	$-(1+x)^{-2}$	$2(1+x)^{-3}$	$-3!\cdot(1+x)^{-4}$	$4!\cdot(1+x)^{-5}$	$\cdots$

Putting in the value $x=0$ gives the necessary values for the Maclaurin series.

$f(x)$	$f(0)$	$f'(0)$	$f''(0)$	$f'''(0)$	$f^{(4)}(0)$	$f^{(5)}(0)$	$\cdots$
e^x	1	1	1	1	1	1	$\cdots$
$\sin x$	0	1	0	-1	0	1	$\cdots$
$\cos x$	1	0	-1	0	1	0	$\cdots$
$\log(1+x)$	0	1	-1	2	$-3!$	$4!$	$\cdots$

Now, putting these coefficients into the definition, we obtain the Maclaurin series for these four functions.

$$\begin{aligned}
e^x &\mapsto 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \\
\sin x &\mapsto x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \\
\cos x &\mapsto 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots \\
\log(1+x) &\mapsto x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots
\end{aligned}$$

Note that the denominators in the series for $\log(1+x)$ are of the form $\frac{1}{k}$, rather than $\frac{1}{k!}$, since the value $f^{(k)}(0)$ is $(k-1)!$ and cancels all but the k.

4 Convergence of Taylor series

A Taylor series is a sum of an infinite number of terms. As such, it may or may not converge. The next lecture will begin a careful study of the convergence of series, but for now we just mention a simple criterion.

The Taylor series of f around a center c converges to the value $f(x)$ at a particular value of x if and only if the remainder $R_n(x)$ goes to 0 as n goes to infinity.

Here, $R_n(x)$ is the remainder term discussed in the previous lecture; it is equal to $f(x) - P_n(x)$. Observe that since we have an upper bound $\frac{\mathfrak{M}_{n+1}}{(n+1)!}|x-c|^{n+1}$ for $|R_n(x)|$, we can use this upper bound to show that a Taylor series actually converges to the function that it is meant to approximate.

Example 4.1. Consider the function $f(x) = e^x$, whose Maclaurin series is $1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$. Suppose that x is some fixed value greater than 1. Then for any n, the n^{th} derivative of e^x is also e^x, and the maximum value of this on $[0, x]$ is e^x. Therefore $\mathfrak{M}_{n+1} = e^x$ in this context. From the Taylor remainder theorem, it follows that for each n,

$$|R_n(x)| \leq \frac{e^x}{(n+1)!}x^{n+1}.$$

Now, as n grows very large, the value $(n+1)!$ grows much faster that the value x^{n+1}, so this upper bound goes to 0 (a more precise argument can be given by comparing these error terms to a geometric sequence, which we will discuss later). Therefore the function e^x is precisely equal to its Maclaurin series for all positive values of x. A similar argument shows equality for negative values of x as well.

$$\begin{aligned}
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}x^k
\end{aligned}$$

Example 4.2. Consider the function $f(x) = \sin x$. The derivatives at $x = 0$ of $\sin x$ follow the regular pattern: $0, 1, 0, -1, 0, 1, 0, -1, \ldots$. Therefore the Taylor series is easy to write down.

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$$

Now, all derivates of $\sin x$ are either $\pm \sin x$ or $\pm \cos x$; in particular, the values of all derivatives are bounded by 1 at all places. It follows that for all n, the remainder of the Taylor approximation around 0 is bounded as follows.

$$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$$

By reasoning similarly to before, this upper bound goes to 0 as n goes to infinity, so in fact the Taylor series converges to $\sin x$ for all value of x.

$$\begin{aligned}\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}\end{aligned}$$

In fact the exact same reasoning will apply to $\cos x$; it is also equal to its Maclauring series at all points x.

$$\begin{aligned}\cos x &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}\end{aligned}$$

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5 A series expression for π

Notice that since we know that the function e^x is exactly equal to its Maclaurin series, it is possible to calculate it to arbitrary accuracy by summing rational numbers.

$$e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!} + \cdots$$

Indeed, this infinite sum is sometimes even used as a definition for the number e.

There are other transcendental numbers that can be calculated using rather simply infinite series. For example, suppose that we want to know the value of π. One way to access it (by no means the best way, but a way nonetheless) is using the fact that:

$$\frac{\pi}{4} = \tan^{-1}(1).$$

How is this useful? Well, given that we understand the function $\tan^{-1}$ fairly well at the value $x = 0$, we can use this knowledge in order to calculate other values of the function, using its Maclaurin series. There is actually a clever way to find this series.

For the rest of this section, we will not justify each step (e.g. verify that everything converges), but merely press forward to the goal. We will see, moving forward, how to justify some steps like these.

Consider the fact that:

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

Combine this with the following Taylor series (which could be computed separately, or which follows from differentiating the Maclaurin series for $\ln(1+x)$ computed earlier).

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots ,$$

Substituting x^2 in for x in the second series and integrating:

$$
\begin{aligned}
\frac{1}{1+x^2} &= 1 - x^2 + (x^2)^2 - (x^3)^3 + \cdots \\
&= 1 - x^2 + x^4 - x^6 + \cdots \\
\tan^{-1} x + C &= \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx \\
&= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots
\end{aligned}
$$

Now, by considering the value $x = 0$, we can conclude that this constant C should be 0, so in fact the Maclaurin series for $\tan^{-1} x$ is simply:

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

In particular, putting in the value $x = 1$, we obtain the following.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This gives one way to calculate π by hand, if necessary. However, there are certainly only ways which converge much more rapidly, but we will not discuss them.