Math 1B, lecture 14: Taylor's Theorem

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1 Introduction

Taylor polynomials give a convenient way to describe the local behavior of a function, by encapsulating its first several derivatives at a point. The extent to which they are useful as approximations is governed by studying the error, or remainder, when the values of Taylor polynomials are used in place of the exact value of a function. The main result in this study is Taylor's theorem, which is a consequence of (and a generalization of) the mean value theorem.

Taylor's theorem is the most important theorem is differential calculus. Qualitatively, it answers the following question: to what extent do the derivatives of a function at a single point dictate the behavior of the function at nearby points? Given that differential calculus is entirely founded on the idea that derivatives capture, in a sequence of numbers, the way a function is moving near a point, such a theorem is necessary to make precise what, exactly, is meant by this.

Quantitatively, of course, Taylor's theorem answers a practical concern: if we wish to use Taylor approximation to actually approximate a function, how much accuracy can we assure ourselves in this approximation? This leads naturally into the notion of Taylor series, which are formed by taking Taylor polynomials of larger and larger degree, thus forming an infinite sum. These will be studied next time.

The reading for today is Gottlieb §30.2. The homework is problem set 13 and a topic outline.

2 The mean value theorem

The derivative of a function is meant to describe the slope of that function. The main theorem which demonstrates that this analogy is apt is the mean value theorem, which is the first main theorem of differential calculus. We recall it here before generalizing it.

Theorem 2.1. *Let f be a differentiable function. For any a, b, the quantity $\frac{f(b)-f(a)}{b-a}$, which measures the slope of the line from the point $(a, f(a))$ to $(b, f(b))$, is equal to $f'(c)$, where c is some number between a and b.*

In other words, the slope of a line between two points on the graph of $y = f(x)$ is equal to the derivative of $f(x)$ somewhere between the two points. The theorem does not make any guarantees about where, however.

The following image shows an homage to the mean value theorem by the government in Beijing.

Another way to express the equation in the mean value theorem is the following.

$$f(b) = f(a) + f'(c)(b - a)$$

This form is the form from which it will be generalized to give Taylor's theorem. It may be thought of as saying that $f(b)$ is close to $f(a)$, with the difference governed by the distance between b and a (equal to $b - a$) and the derivative of f between a and b.

3 Taylor's theorem

Let f be a function, and c some value of x (the "center"). Denote, as usual, the degree n Taylor approximation of f with center $x = c$ by $P_n(x)$. As discussed before, this is the unique polynomial of degree n (or less) that matches $f(x)$ and its first n derivatives at $x = c$. It is given by the expression below.

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

Taylor's theorem states that the difference between $P_n(x)$ and $f(x)$ at some point x (other than c) is governed by the distance from x to c and by the $(n + 1)^{st}$ derivative of f. More precisely, here is the statement.

Theorem 3.1 (Taylor's theorem)**.** *Assume that f is $(n + 1)$-times differentiable, and P_n is the degree n Taylor approximation of f with center c. Then if x is any other value, there exists some value b between c and x such that*

$$f(x) = P_n(x) + \frac{f^{(n+1)}(b)}{(n + 1)!}(x - c)^{n+1}$$

Observe that this "remainder" term $\frac{f^{(n+1)}(b)}{(n+1)!}(x - c)^{n+1}$ looks quite a lot like the next term in the next Taylor approximation of $f(x)$. The difference is that $f^{(n+1)}$ is evaluated at *some* point, not necessarily the center, and we do not know what point it might be.

Also observe that in the case $n = 0$, Taylor's theorem is precisely the mean value theorem. In fact, the mean value theorem is the main tool for proving Taylor's theorem, as will be demonstrated in the appendix.

This theorem is usually used in the following form, which follows immediately from it.

Corollary 3.2. *Let $\mathfrak{M}_{n+1}$ be the maximum of $|f^{(n+1)}(x)|$ on some interval containing the center c*[1]*. Then for any x in this interval, the error of the Taylor approximation is bounded as follows.*

$$|f(x) - P_n(x)| \leq \frac{\mathfrak{M}_{n+1}}{(n+1)!}|x-c|^{n+1}$$

Observation 3.3. In particular, if $f^{(n+1)}(x) = 0$, then the degree n Taylor approximation is exactly correct. Of course, this is not surprising: if the $(n+1)^{st}$ derivative is 0 as a function, then $f(x)$ is just a polynomial of degree at most n, so the Taylor approximation will be exactly correct.

Observation 3.4. This error bound is formally quite similar to the error bounds for numerical approximations that we studied earlier. Recall, for example, that the error for the trapezoid approximation on an interval $[a, b]$ is bounded by the following expression.

$$\left| \int_a^b f(x)dx - T_n \right| \leq \frac{\mathfrak{M}_2}{12n^2}(b-a)^3$$

In the case of both of these error bounds, if the approximation takes n^{th} order behavior into account, then its error is governed by the size of the $(n+1)^{st}$ derivative and some power of the length of the interval being considered. Of course, it is important to remember that one of these is approximating an integral, and the other is approximation a function itself. Both, however, share the trait that they approximate by assuming that the function is a polynomial of a certain degree, and computing what the answer would be if this were the case.

4 Examples

We will now provide error bounds for the some of the example functions considered in the previous lecture.

Example 4.1. Approximate $\sqrt{5}$ to at least accuracy $\frac{1}{100}$.
In previous lecture, we showed that the Taylor approximation of $f(x) = \sqrt{x}$, with center $x = 4$, is given by:

$$P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2.$$

Recall that the center $x = 4$ was chosen because the square root of 4 is an integer, and it is very easy to work out all derivatives of the function $\sqrt{x}$ by hand at this point.

Now, this gives the approximation $P_2(5) = 2\frac{15}{64} = 2.234375$.

The third derivative of $\sqrt{x}$ is $f'''(x) = \frac{3}{8}x^{-5/2}$. The maximum value of this function in the interval $[4, 5]$ is attained at $x = 4$ (since it is a positive decreasing function), where it is equal to $\frac{3}{8} \cdot \frac{1}{32} = \frac{3}{256}$. So $\mathfrak{M}_3 = \frac{3}{256}$. By corollary 3.2,

$$\begin{aligned} \left| \sqrt{5} - 2\frac{15}{64} \right| &\leq \frac{3}{256}\frac{1}{3!}(5-4)^3 \\ &\leq \frac{1}{512} \end{aligned}$$

Therefore, the approximation $2\frac{15}{64}$ is guaranteed to be accurate to within at least $\frac{1}{512}$, which is well less than $\frac{1}{100}$.

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[1]The textbook and handouts use simply the symbol M for this quantity. In these notes I use $\mathfrak{M}_{n+1}$ since this is the same notation that was used in Lecture 4.

Observation 4.2. Notice that in the example above, the error bound was exactly equal to what the next term would have been, if we had used the next Taylor approximation $P_3(x)$. This will happen in a wide range of cases, as will become clear as we proceed. In fact, it should be clear that it is true whenever $f^{(n+1)}(x)$ is maximized at $x = c$ itself.

Example 4.3. Consider $f(x) = e^x$. As discussed before, the degree n Taylor approximation is as follows.

$$\begin{aligned} P_n(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n \\ &= \sum_{k=0}^{n} \frac{1}{k!}x^k \end{aligned}$$

Now, the next derivative is $f^{(n+1)}(x) = e^x$. The maximum value of this on $[0, x]$ is exactly e^x. Therefore, by corollary 3.2, using $\mathfrak{M}_{n+1} = e^x$,

$$|e^x - P_n(x)| \leq \frac{e^x}{(n+1)!}x^{n+1}$$

Notice that, as a fraction of the correct answer e^x, the error is at most $x^{n+1}/(n+1)!$. For very large n, this becomes very small (we will discuss this in more detail soon).

Example 4.4. Suppose that $f(x) = \sin x$ or $f(x) = \cos x$. Then all derivatives of either function are either $\pm \sin x$ or $\pm \cos x$. In particular, all derivates are bounded, at all points, simply by 1. Therefore is corollary 3.2, we know that $\mathfrak{M}_{n+1} \leq 1$ in all cases. Therefore, for all x and all n,

$$\begin{aligned} |\sin x - P_n(x)| &\leq \frac{1}{(n+1)!}x^{n+1} \\ |\cos x - P_n(x)| &\leq \frac{1}{(n+1)!}x^{n+1} \end{aligned}$$

Of course, the P_n in the first inequality is the Taylor approximation for $\sin x$, whereas the P_n in the second inequality is the Taylor approximation for $\cos x$.

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5 Appendix: Proof of Taylor's theorem

The proof of Taylor's theorem is actually quite straightforward from the mean value theorem, so I wish to present it. However, it involves enough notation that it would be difficult to present it in class.

First, the following lemma is a direct application of the mean value theorem.

Lemma 5.1. *Suppose that $f(x)$ is some function that is differentiable $(n+1)$ times. Suppose also that $f(c) = 0,\ f'(c) = 0,\ \ldots, f^{(n)}(c) = 0$, and that $f(x) = 0$ for some x other than c. Then there exists some b between c and x such that $f^{(n+1)}(b) = 0$.*

Proof. Since $f(c) = 0$ and $f(x) = 0$, the mean value theorem shows that there is some b_1 between c and x such that $f'(b_1) = 0$. Now, since this $f'(c) = 0$ and $f'(b_1) = 0$, the mean value theorem shows that there is some b_2 between c and b_1 such that $f''(b_2) = 0$. Continue in this fashion, using the equations at c to construct a sequence of values $b_3, b_4, \ldots b_n$. Eventually, this shows that there is some b_n between c and x such that $f^{(n+1)}(b_n) = 0$. Now $b = b_n$ is the desired value. □

Although it may not appear that this lemma is related, we can use it to complete the proof of Taylor's theorem, as follows.

Let $P_n(x)$ be the degree n Taylor approximations of $f(x)$ with center $x = c$. Then the function $g(x) = f(x) - P_n(x)$ has the property that $g(c) = 0$, and also $g'(c) = g''(c) = \cdots = g^{(n)}(c) = 0$. In effect, we have constructed P_n specifically to make these equations true.

Now let y be some value other than c. The error $f(y) - P_n(y)$ of the Taylor approximation at y is simply $g(y)$. Define a constant $C = g(y)/(y-c)^{n+1}$. Now define another function, $h(x) = g(x) - C(x-c)^{n+1}$. The function h has been constructed specifically to have the properties described in the lemma.

$$\begin{aligned} h(c) &= 0 \\ h'(c) &= 0 \\ h''(c) &= 0 \\ &\cdots \\ h^{(n)}(c) &= 0 \\ h(y) &= 0 \end{aligned}$$

It follows from this that for some value b between c and y, it is the case that $h^{(n+1)}(b) = 0$. But now observe that $h^{(n+1)}(x) = f^{(n+1)}(x) - (n+1)!C$, by an easy calculation from the definitions. Therefore, at $x = b$, we have $h^{(n+1)}(b) = f^{(n+1)}(b) - (n+1)!C = 0$, so $C = \frac{f^{(n+1)}(b)}{(n+1)!}$. But from the definition of C, it follows that $\frac{f^{(n+1)}(b)}{(n+1)!} = \frac{g(y)}{(y-c)^{n+1}}$, so $g(y) = \frac{f^{(n+1)}(b)}{(n+1)!}(y-c)^{n+1}$. Therefore

$$f(y) = P_n(y) + \frac{f^{(n+1)}(b)}{(n+1)!}(y-c)^{n+1}.$$

This is precisely Taylor's theorem.