Math 1B, lecture 11: Improper integrals I

Nathan Pflueger

30 September 2011

1 Introduction

An improper integral is an integral that is unbounded in one of two ways: either the domain of integration is infinite, or the function approaches infinity in the domain (or both). Conceptually, they are no different from ordinary integrals (and indeed, for many students including myself, it is not at all clear at first why they have a special name and are treated as a separate topic). Like any other integral, an improper integral computes the area underneath a curve. The difference is foundational: if integrals are defined in terms of Riemann sums which divide the domain of integration into n equal pieces, what are we to make of an integral with an infinite domain of integration?

The purpose of these next two lecture will be to settle on the appropriate foundation for speaking of improper integrals. Today we consider integrals with infinite domains of integration; next time we will consider functions which approach infinity in the domain of integration. An important observation is that not all improper integrals can be considered to have meaningful values. These will be said to diverge, while the improper integrals with coherent values will be said to converge. Much of our work will consist in understanding which improper integrals converge or diverge. Indeed, in many cases for this course, we will only ask the question: "does this improper integral converge or diverge?"

The reading for today is the first part of Gottlieb §29.4 (up to the top of page 907). The homework is problem set 10 (which includes weekly problem $9, 10$, and 11) and a topic outline.

2 Example: escape energy

This example makes use of some basic principles of physics. These are not part of the course, so don't worry if they are not familiar.

In Newtonian physics, one sometimes speaks of the "escape energy" of an object in a gravitational field (say, near a planet). This is the amount of energy necessary for this object to escape from the gravitational pull of a the planet. As an example, we demonstrate how to calculate escape energy.

Suppose that a spacecraft begins at a distance r_0 from a planet, and it wishes to escape the gravitational field of the planet.

According to Newton, the gravitational force exerted on an spacecraft of mass m by a planet of mass M that is a distance r away is given by GMm/r^2, where G is a universal constant. It follows from this that the amount of energy needed for the craft to move from a distance r_0 to a distance b away from the planet is given as follows.[1]

[1] What is at work here is the basic principle that work is equal to force times distance. The quantity GMm/r^2 is the force exerted by the planet, and dr represents a very small distance; summing these up we obtain the total work necessary to move from one point to another, against the force of the planet's gravity.

$$\begin{aligned}
\text{(energy to travel to distance } b) &= \int_{r_0}^{b} \frac{GMm}{r^2} dr \\
&= GMm \int_{r_0}^{b} \frac{1}{r^2} dr \\
&= GMm \left(\frac{1}{r_0} - \frac{1}{b} \right)
\end{aligned}$$

So how much energy does it take to escape from the gravitational field? From one perspective, this question doesn't make any sense. No matter how far the spacecraft travels from the planet, the planet will always exert a little bit of force. But this bit of force becomes very small eventually. So here is the usual notion: *the escape energy is the amount of energy needed for the spacecraft to travel infinitely far away from the planet.* In other words, this should be upper bound of the energy needed to travel to any particular distance b. In this case, the desired figure is obtained by noting that the term $\frac{1}{b}$ goes to 0 as b goes to infinity. Therefore:

$$\begin{aligned}
\text{(escape energy)} &= \lim_{b \to \infty} GMm \left(\frac{1}{r_0} - \frac{1}{b} \right) \\
&= GMm/r_0.
\end{aligned}$$

We could also write this with the following notation, which will be our notation for improper integrals.

$$\text{(escape energy)} = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr$$

This notation is meant to suggest that the expression on the right signifies the amount of energy needed to travel, not to any finite point, but *all the way to infinity.* We shall now formalize what we mean by this notation in general.

3 Definitions

We wish to make a definition that will make sense of the following notation.

$$\int_{a}^{\infty} f(x) dx$$

Informally speaking, we already know exactly what this means: it means *the signed area under the curve $y = f(x)$ from $x = a$ to ∞.* What need do we have for any more definition? Well, just as we needed to discuss Riemann sums in order to be less vague about what we meant by signed area at all, we need to be less vague about what we mean by a signed area that reaches out into unbounded reaches.[2]

So let us formalize this signed area. Our current working definition of the integral by means of Riemann sums cannot possibly work for infinite intervals, since it requires that the domain of integral be split into n equal pieces. What we must do is make a new definition in terms of our old definition. Given that $\int_a^b f(x)dx$ describes the area under part of the curve (the part from $x = a$ to $x = b$), and that we want all of the area, we should just take b larger and larger, taking into account more and more of the curve. To get all of the area, we should take this process all the way to infinity; that is, we should take a limit. Here is the definition.

[2]Or we may have no such need. It is worth noting that Bernhard Riemann (1826-1866) was born nearly one hundred years after the death of Sir Isaac Newton (1642-1727), during which time scientists got by, happily computing integrals without Riemann's rigorous definition. Indeed, there was *no formal definition of a limit at all* until the early 19th century. There is a strong argument to be made that, for the student of mathematics, these technical definitions are not helpful and could be deferred until the notions are mastered on an intuitive level. Regardless, this approach is not how this course is structured.

Definition 3.1. The improper integral of a function over a half-infinite interval is

$$\begin{aligned} \int_a^\infty f(x)dx &= \lim_{b\to\infty}\int_a^b f(x)dx \\ &\text{or} \\ \int_{-\infty}^a f(x)dx &= \lim_{b\to-\infty}\int_b^a f(x)dx \end{aligned}$$

whenever this limit exists. If the limit exists, the integral is said to *converge*. If the limit does exist, the integral is said to *diverge*.

It is important to note that in some cases, there is no meaningful value for an improper integer. Probably the simplest example is $\int_a^\infty dx$, which could not equal anything but ∞. This is why it is necessary to first consider whether the integral converges.

One final definition is needed, to account for the case where we wish to integral a function across the entire number line.

Definition 3.2. The improper integral of a function across the entire number line is

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx,$$

where a is any number, whenever both of the integrals on the right converge. If either one diverges, the integral on the left is said to *diverge*. Otherwise, it is said to *converge*.

You should convince yourself that this definition does not depend on which value of a is chosen. In practice, any convenient value a will work.

It is worthwhile to think about why a definition such as $\int_{-\infty}^\infty f(x)dx = \lim_{b\to\infty}\int_{-b}^b f(x)dx$ will not do. This will be the subject of a homework problem.

I end this section by noting that in higher mathematics, there is a much nicer, cleaner definition of integral which eliminates the need for all this mucking around with limits and also makes integration well-defined in a much wider set of situations. This theory is called *Lebesgue integration*, and we will certainly not discuss it in this course.

4 Examples

We now discuss several examples of improper integrals.

Example 4.1. $\displaystyle\int_1^\infty \frac{1}{x^2}dx.$

This was basically our example from the previous section. By computing the antiderivative, we can see that this integral converges, and compute it.

$$\begin{aligned} \int_1^\infty \frac{1}{x^2}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx \\ &= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b \\ &= \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) \\ &= 1 \end{aligned}$$

Example 4.2. $\displaystyle\int_1^\infty \frac{1}{x} dx$.
Here again, we can compute the antiderivative.

$$\begin{aligned}\int_1^\infty \frac{1}{x} dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x} dx \\ &= \lim_{b\to\infty} [\ln|x|]_1^b \\ &= \lim_{b\to\infty} (\ln|b| - 0)\end{aligned}$$

But this expression clearly goes to ∞ (slowly) as b goes to infinity. So this integral **diverges**. There is an infinite amount of area underneath the curve $y = 1/x$, even if it doesn't seem to look that way. It is worth pondering how this could be possible.[3]

Example 4.3. $\displaystyle\int_0^\infty \cos x dx$.
Begin as before.

$$\begin{aligned}\int_0^\infty \cos x dx &= \lim_{b\to\infty} \int_0^b \cos x dx \\ &= \lim_{b\to\infty} [\sin x]_0^b \\ &= \lim_{b\to\infty} \sin b\end{aligned}$$

So this integral **diverges** as well. Note, however, that it diverges for a different reason that the previous integral did: in this case, the area under the curve oscillated back and forth and never converges to a single value, but it does not ever explode to infinity.

---

5 The comparison test

There is a convenient test that will tell, in many situations, whether an integral converges or not. We will not give a proof of this fact, but we have given some intuitive arguments for it in the examples that follow.

Theorem 5.1 (The comparison test)**.** *Suppose that $f(x)$ and $g(x)$ are two functions such that $f(x) \geq g(x) \geq 0$ for all x in $[a, \infty]$. Then:*

- *If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges, and*
- *If $\int_a^\infty g(x)$ diverges, then $\int_a^\infty f(x)dx$ diverges.*

Notice that the second bullet point is redundant, in that it is logically equivalent to the first (in logical jargon, the two bullet points are contrapositives of each other).

It is crucial to note that *both functions must be nonnegative everywhere for the comparison test to apply as stated*[4].

This test is useful in situations where it may be hard to evaluate the antiderivative of a function, but it is very easy to compare the function to simpler-looking function. If you wish to show that an integral diverges, just show that the function is larger than another function that diverges. Similarly, if the function is bounded above by a function whose integral converges, then its integral certainly must converge as well.

[3] A very much related, and historically earlier, conundrum, is the so-called *harmonic series*, which is the expression $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$. Although the terms go to 0, this sum diverges to infinity. We will discuss this and other, related, series later in the course.

[4] Actually, it is not hard to generalize the test so that it also applies if the inequalities in the statement hold, not for all $x \geq a$, but simply for all $x \geq c$, where c is some number. But the non negativity assumption is still critical.

6 More examples

These examples try to make use of the comparison test.

Example 6.1. Does $\displaystyle\int_1^\infty e^{-x^2}dx$ converge?

And we return, as we have perennially in this course to an old white whale: the integral of e^{-x^2}. Unfortunately, we will never find the antiderivative of this function. But can we at least conclude whether its integral from 0 to ∞ is convergent or not? It is not clear how to proceed, given that until now, we have always computed the antiderivative explicitly.

A bit of wishful thinking does the trick. We can't integrate e^{-x^2}, but we can integrate xe^{-x^2}, by substitution.

$$\begin{aligned}
\int_1^\infty xe^{-x^2}dx &= \lim_{b\to\infty}\int_1^b xe^{-x^2}dx \\
&\quad u = x^2,\ du = 2xdx \\
&= \lim_{b\to\infty}\int_1^{b^2}\frac{1}{2}e^{-u}du \\
&= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-u}\right]_1^{b^2} \\
&= \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2e}\right) \\
&= \frac{1}{2e}
\end{aligned}$$

So the integral of xe^{-x^2} does converge. Does this help? It turns out, yes. For all $x \geq 0$, $xe^{-x^2} \geq e^{-x^2}$. So the area under the curve $y = e^{-x^2}$ cannot exceed the area under the curve xe^{-x^2}. Both are positive functions, so there are no technicalities about signed area. So this implies that in fact the original integral $\displaystyle\int_1^\infty e^{-x^2}dx$ also **converges**.

We may more succinctly (and rigorously) state this argument as follows: since $xe^{-x^2} \geq e^{-x^2} \geq 0$ for all $x \geq 1$, and $\int_1^\infty xe^{-x^2}$ converges , it follows from the comparison test that $\int_1^\infty e^{-x^2}dx$ also converges.

An appendix to these notes computes the integral $\displaystyle\int_{-\infty}^\infty e^{-x^2}dx$, which is a rather famous improper integral, since it can be computed without knowing the antiderivative of the integrand, by a rather clever trick.

Example 6.2. Does $\displaystyle\int_2^\infty \frac{x^2+1}{x^3-1}dx$ converge?

In principle, we could compute the antiderivative of this function by means of partial fractions. But for purposes of the convergence question, let us punt on this and try to use the same method as the previous example: compare to another function. Which function should we compare to? Here is a heuristic argument: for large x, the only terms that matter are the leading terms, so the integrand is approximately $\frac{x^2}{x^3} = \frac{1}{x}$. Can we compare to $\frac{1}{x}$? It turns out that we can. Observe that for all $x \geq 2$, $x^2+1 \geq x^2$ and $x^3-1 \leq x^3$, hence $\frac{1}{x^3-1} \geq \frac{1}{x^3}$. All of these terms are positive for $x \geq 2$, so we can multiply these two inequalities to obtain $\frac{x^2+1}{x^3-1} \geq \frac{x^2}{x^3} = \frac{1}{x}$ for all $x \geq 2$. From example 4.2, the integral $\int_2^\infty \frac{1}{x}dx$ diverges (actually, we showed that the integral from 1 to ∞ diverges, but the same argument works). Therefore, we can apply the comparison test (described in the next section); since $\frac{x^2+1}{x^3-1} \geq \frac{1}{x} \geq 0$ for all $x \geq 2$, and $\int_2^\infty \frac{1}{x}dx$ diverges, the integral

$\int_2^\infty \frac{x^2+1}{x^3-1}dx$ also **diverges**.

Example 6.3. Does $\displaystyle\int_1^\infty \frac{\sin x}{x}dx$ converge?

It is tempting to attempt to using the comparison test with $\frac{1}{x}$, but the method does not apply here: this function alternates between being positive and negative, so the comparison test cannot be applied. So, with our present methods, the answer is **inconclusive**. In fact, this integral does converge, but we will most likely not show this in the course. We will have the necessary methods to establish this convergence, so I may write it up as an appendix.

---

7 Appendix: A famous improper integral

It has come up several times in this course that there is no nicely-stated antiderivative of the function e^{-x^2}. Remarkably, though, the improper integral $\int_{-\infty}^\infty e^{-x^2}dx$ can be evaluated exactly, and has a rather surprising value. That this integral converges is not too hard to show using the comparison test, although I omit these details for now.

We will be a bit fast and loose with definitions in this appendix, and will make use, implicitly, of some advanced ideas. In my defense, I will say only that I promise that there is no funny business, and that every time I wave my hands, there is something precise that could be said instead. Besides, I believe that it is always helpful to see sophisticated ideas (in this case, ideas from multivariable calculus) in simple cases long in advance of trying to study them systematically.

Let I denote the integral that we are seeking.

$$I = \int_{-\infty}^\infty e^{-x^2}dx$$

The ingenious idea that allows this integral to be computed is this: *square the integral, and interpret it as a density problem in the plane.* As far as I know, there is no intuitive way to see this. It really is a stroke of genius. So let us write the square of the integral as the product of two integrals, one in terms of y and one in terms of x.

$$I^2 = \int_\infty^\infty e^{-x^2}dx \int_{-\infty}^\infty e^{-y^2}dy$$

Now, here is how we can interpret this as a density problem in the plane. Suppose that some powder is spread over the entire plane, such that the density at the point (x, y) is given by $e^{-x^2-y^2}$. What is the total amount of power? I claim that it is equal to I^2. Indeed, suppose we compute the amount of powder in a thin strip near the vertical line $x = c$ of thickness Δx. This amount should be $e^{-x^2}\int_\infty^\infty e^{-y^2}dy \cdot \Delta x$. Therefore:

$$\begin{aligned}
\text{(Total powder)} &= \int_\infty^\infty e^{-x^2}\left(\int_{-\infty}^\infty e^{-y^2}dy\right)dx \\
&= \int_\infty^\infty e^{-x^2}dx \cdot \int_{-\infty}^\infty e^{-y^2}dy \\
&= I^2
\end{aligned}$$

It may appear as if all we have done is posed another, just as difficult problem. But in fact, something miraculous has occurred. We can compute the amount of powder in the plane in a separate way, *by slicing the plane differently.* How should we slice it? Well, notice that the powder density is $e^{-x^2-y^2}$, which is the same thing as e^{-r^2}, where r is the distance to the origin. So let us slice the plane using concentric circles. By the usual argument: a slice of width Δr near radius r should have area approximately $2\pi r\Delta r$. Therefore, we can compute the total powder, also known as I^2, as follows.

$$
\begin{aligned}
I^2 &= \int_0^\infty 2\pi r e^{-r^2} dr \\
& \quad u = r^2,\ du = 2rdr \\
&= \int_0^\infty \pi e^{-u} du \\
&= \pi \lim_{b\to\infty} \left(1 - e^{-b}\right) \\
&= \pi
\end{aligned}
$$

Now, since I is positive, we can conclude that it is equal to the square root of π. We obtain the famous result:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$