# Lecture 4: Continuity I

Nathan Pflueger

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## 1 Introduction

We meet many different sorts of functions in calculus and real-world data, some better-behaved than others. The main tools of calculus apply primarily to functions whose graphs you can draw without lifting your pen. There is a precise mathematical notion of such a function: it is called a *continuous function.*

Continuous functions play two principle roles in calculus. First, when modeling real-world data, we often want to fit a continuous function to data that are not continuous to begin with. This allows more powerful mathematical techniques; it often also unearths deeper unifying structure in the data. Second, continuous functions are used as "props" to which we compare the data we have, in order to describe it's behavior.

This lecture describes the mathematical notion of continuity, and then illustrates these two roles. We examine how continuous functions can be cooked up to model situations, and we use them as "props" to show how to evaluate some otherwise difficult limits using a technique called the "squeeze theorem."[1]

The reference for today is Stewart §2.4.

## 2 Continuous functions

Some functions are nicer than others. In calculus, it is often necessary to focus attention on functions that are well-behaved enough for mathematical techniques to work well. One of the basic restrictions we often impose on functions is that they should br continuous.

A common informal description of a continuous function is "you can draw its graph without lifting your pen." Let's try to pin down what, more precisely, this could mean. Why would you need to lift your pen? Presumably because the graph jumps to someplace else. What does it mean to jumps? It means that whatever you've just finished drawing didn't lead to the next point on the graph. That is, the nearby values didn't approach the new value. How can we state this mathematically? The usual approach is to speak in terms of limits.

Remember that the function value $f(0)$ is not necessarily equal to the limit $\lim_{x \to 0} f(x)$ (indeed, the limit may not even exist). When it does, though, this means that the nearby values determine the value f(0), and lead up to it. Essentially, you wouldn't need to lift your pen as you come into the value $x = 0$.

**Definition 2.1.** If $c$ is a number such that $f(c) = \lim_{x \to c} f(x)$, then we say that the function $f$ is *continuous* at $c$.

If either $f(c)$ or $\lim_{x \to c} f(x)$ don't exist or are not defined, we'll also say that $f$ is not continuous at $c$.

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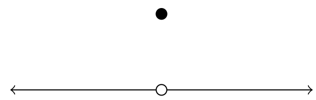
*Example* 2.2. Consider the function $f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$. This function is not continuous at $x = 1$ since the limit is $\lim_{x \to 1} f(x) = 1$ even though $f(1) = 2$. It is continuous at all other points.

[1] When I learned this theorem, it was called the "sandwich theorem." Authors vary on what corny nickname to use; feel free to share your own with me.



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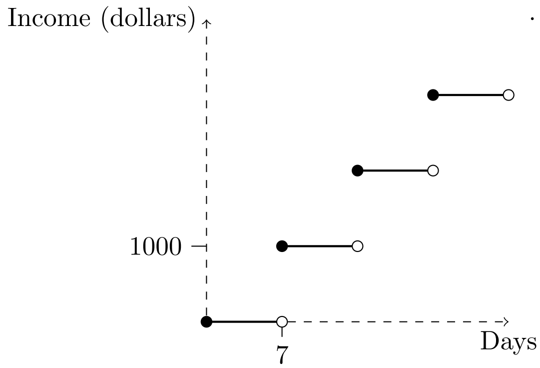
*Example* 2.3. Suppose that you have just begun work at a new job, on a Monday morning. Every Monday after this one, you will be paid \$1000 at the beginning of the day. Then if $f(t)$ is the function that tells how many dollars you have been paid at a time $t$ (measured in days or fractions thereof), the graph of $f(t)$ looks as follows.




This function is discontinuous at all values of $t$ that are precisely multiples of 7.

As an aside: one way to write this function in closed form is $f(t) = 1000\lfloor \frac{t}{7} \rfloor$, where $\lfloor x \rfloor$ denotes the function which rounds $x$ *down* to an integer (the function $f(x) = \lfloor x \rfloor$ is called the *floor function*.).

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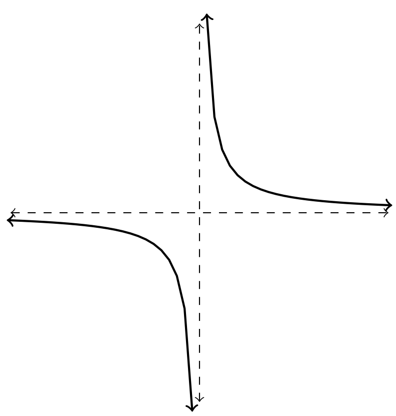
*Example* 2.4. Consider the function $f(x) = \frac{1}{x}$. Where is it continuous?
*Answer:* The function is continuous everywhere it is defined. That is, it is continuous of $(-\infty, 0)$ and $(0, \infty)$. See the picture.



The behavior in the middle is an example of a *vertical asymptote*, which is one of the main types of discontinuity that we will encounter.

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# 3 Approximating with continuous functions

Real-world data often do not follow continuous functions. This can happen for two reasons.

1. The data are not complete, or have errors in them (for example, the bicycle speedometer only knows that time that the wheel completes a revolution; it doesn't know the position of the bike at all times in between).

2. The phenomenon is discontinuous by nature (for example, your income if you are paid a lump sum each Monday morning).

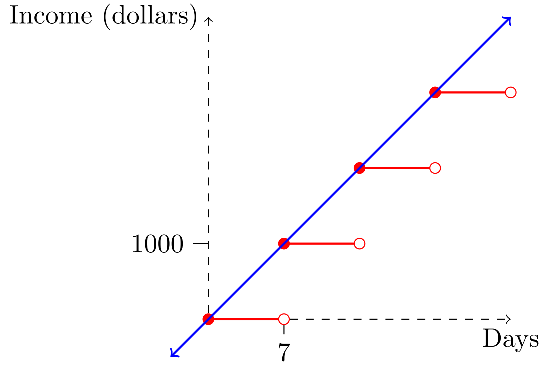
In either case, we can still gain a lot of insight into the problem by giving a suitable continuous approximation. The approximation is often both easier to understand and opens a whole array of tools from calculus.

*Example* 3.1. Consider the "income function" $f(t)$ from example 2.3. As we discussed in that example, this function is not continuous, since it always jumps by multiples of a thousand. But there is a good candidate for a continuous approximation:

$$g(t) = \frac{1000}{7}t$$

The 1000 is because you are paid 1000 dollars at t time; the 7 in the denominator is present because you should divide $t$ (the number of days) by 7 to get the number of weeks that have passed.

Close up, these functions look somewhat different ($f(t)$ is in red, $g(t)$ is in blue)




But when you zoom out to, say, a year, the difference in much less noticeable. The function $g(t)$ will be a more useful approximation for long time frames, since it is much smoother, less complicated, and easier to understand.

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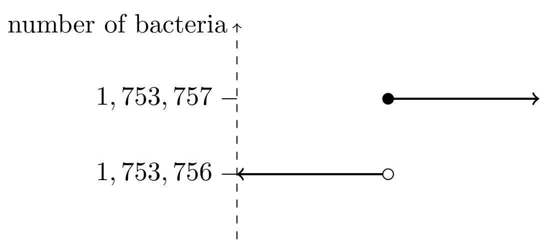
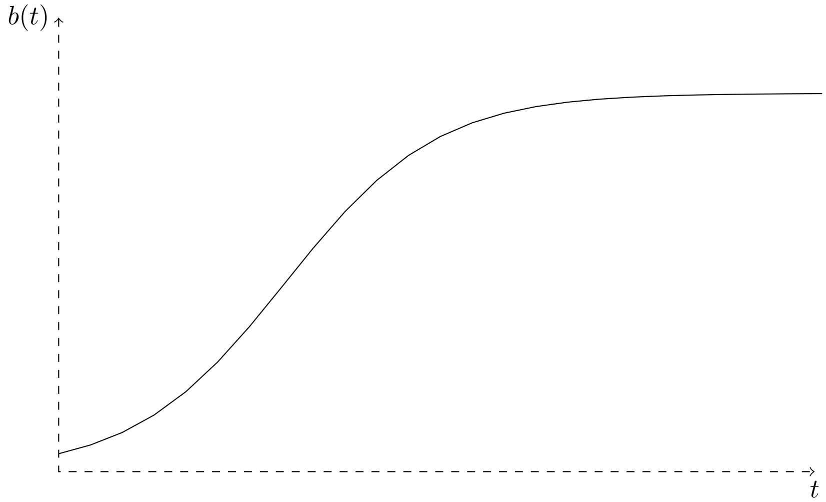
*Example* 3.2. Suppose that there are some bacteria growing in a Petri dish. Let $b(t)$ be the number of bacteria in the dish after $t$ seconds. This function is not continuos (unless it is constant), because it must always "jump" from one whole number value to the next. For example, this might be what the function looks like up close, as one more bacterium is born.[2]




However, you might measure this function by some sort of experiment, and observe that it describes a plot something like the following.

[2]It is probably a philosophical question whether this function ought to be continuous from the right (as I've drawn it) or instead continuous from the left. The question is: does a bacterium cease to not exist at a definite instant, or does it instead begin to exist at that definite moment?




From a distance, this graph looks continuous and rather smooth, so you can attempt to model the curve and apply techniques from calculus to study it (the function I've draw above is called a *logistic function* and is often used to model population growth like in this example).

I should also point out that it is probably not feasible to measure the precise number of bacteria at any instant anyway; the best you can do is measure an approximation, and then in turn fit these approximations to a realistic (continuous) function.

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# 4 Removing discontinuity

Sometimes, slightly modifying the definition of a function can remove discontinuity.

*Example* 4.1. The function $f(x) = \frac{x^2-1}{x-1}$ tells the slope of a secant line from $(1, 0)$ to a point $(x, y)$ on the graph of $y = x^2$. It is not defined (and thus not continuous) at $x = 1$, but it is continuous everywhere else (it is equal to $x + 1$ everywhere else), and its limit at $x = 1$ exists and is equal to 2. So this discontinuity can be removed by just "filling in" this missing point by declaring $f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$.


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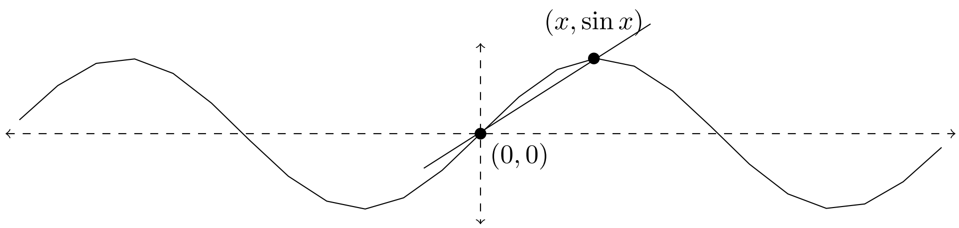
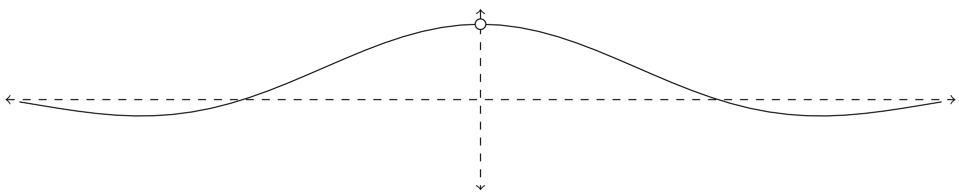
*Example* 4.2. Consider the slope of the tangent line to $y = \sin x$ at $x = 0$. To find it, begin by considering the slope of a secant line from $(0, 0)$ to a nearby point $(x, \sin x)$.




The slope of this line is $\frac{\sin x}{x}$. The function $s(x) = \frac{\sin x}{x}$ is is defined for all values of $x$ except $x = 0$ (where it would be $\frac{0}{0}$). It's graph looks like this.



It is clear from the picture that the singularity is removable, and that the limit is 1. We will return to this example later; the limit $\lim_{x\to 0} \dfrac{\sin x}{x}$ is an important example in the logical structure of calculus. The fact that this limit is *exactly* 1 is (in my opinion) the most important reason why we generally use radians rather than degrees in calculus; I will elaborate on this point later.

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The next example is abstract and has a slightly different form. Here the goal is the cook up a function with certain properties – this sort of alchemy is sometimes useful in constructing models for data you encounter in practice. See if you can think of a physical interpretation of this problem.

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*Example* 4.3. Suppose that a function $f(x)$ has the following form. Here, $m$ and $b$ are constants.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ mx + b & \text{if } x > 1 \end{cases}$$

What must $m$ and $b$ be to ensure that $f(x)$ is continuous?

*Solution.* Regardless of the values of $m$ and $b$, $f(x)$ will be continuous on $(-\infty, 1)$ and $(1, \infty)$, since the functions $y = x^2$ and $y = mx + b$ are both continuous. The other question is about the points $x = 1$. Now,

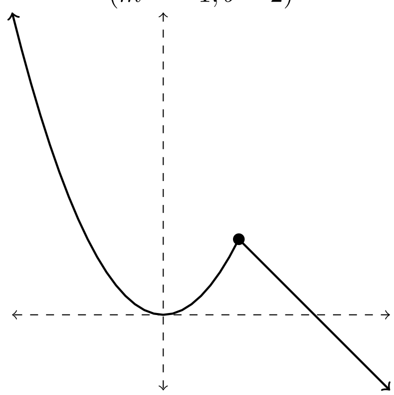
$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} x^2 \\ &= 1^2 \\ &= 1 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (mx + b) \\ &= m \cdot 1 + b \\ &= m + b \end{aligned}$$

Now, $f(x)$ will be continuous at 1 precisely if these two one-sided limits are equal to $f(1)$, which is 1. So this function is continuous precisely if $\boxed{m + b = 1.}$

*Follow-up.* **Not required at this point in the course, but helpful to think about.** What are the possible values of $m$ and $b$ if you want there to be a well-defined tangent line at $x = 1$?

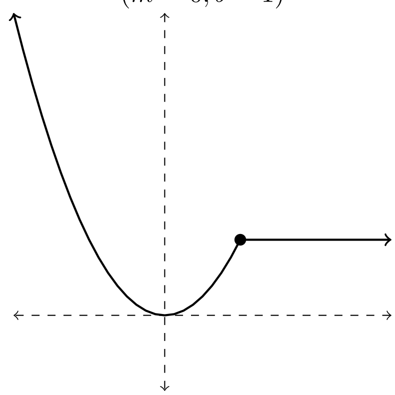
Think about what the graphs of the various possibilities for $f(x)$ look like. The difference is the slope of the line sprouting off from $(1, 1)$. These cases differ by the choice of $m$ and $b$, always chosen so that $m + b = 1$.

$(m = -1, b = 2)$



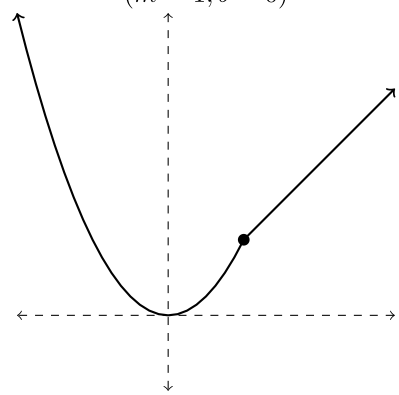
$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ -x+2 & \text{if } x > 1 \end{cases}$$

$(m = 0, b = 1)$



$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

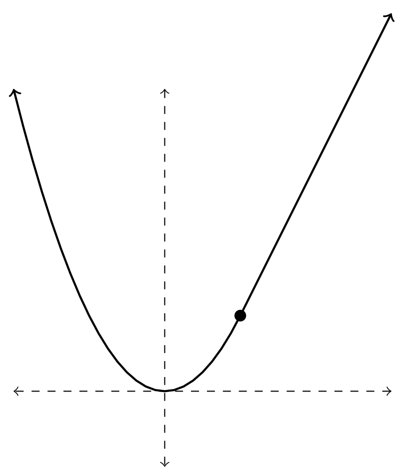
$(m = 1, b = 0)$



$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x & \text{if } x > 1 \end{cases}$$

What are the secant lines from $(1, 1)$ to nearby points $(x, f(x))$? If $x > 1$, the secant line is always the *same* line, namely $y = mx + b$, with slope $m$. Meanwhile, we've seen before than the secant lines for $x < 1$ have slope limiting to $\lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} (x + 1) = 2$ at $x = 1$. So to have a well-defined tangent line, we should simply ensure that $m = 2$.

Thus there is only one possible such function $f(x)$: we must choose $m = 2$ and $b = -1$ to obtain the function $\boxed{f(x) = \begin{cases} x^2 & x \le 1 \\ 2x - 1 & x > 1 \end{cases}}$, whose graph looks as follows.



Note that it differs from the others in that there does not appear to be a sharp "joint" at the point where the line and the parabola have been joined.

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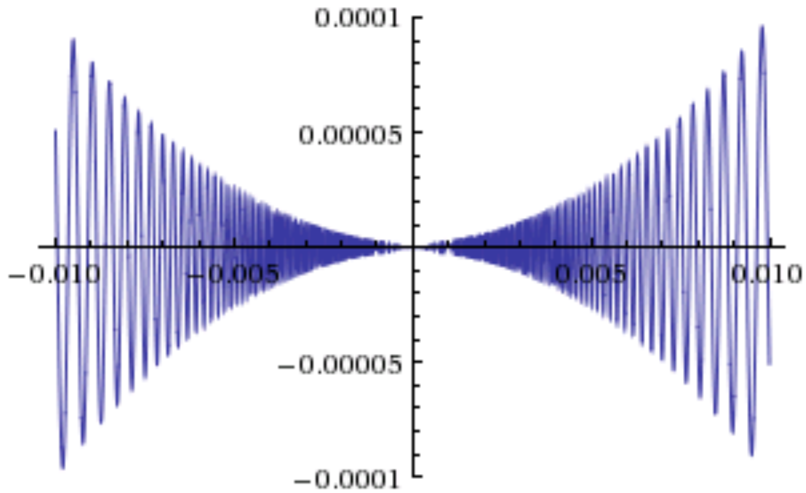
## 5 Evaluating limits: the squeeze theorem

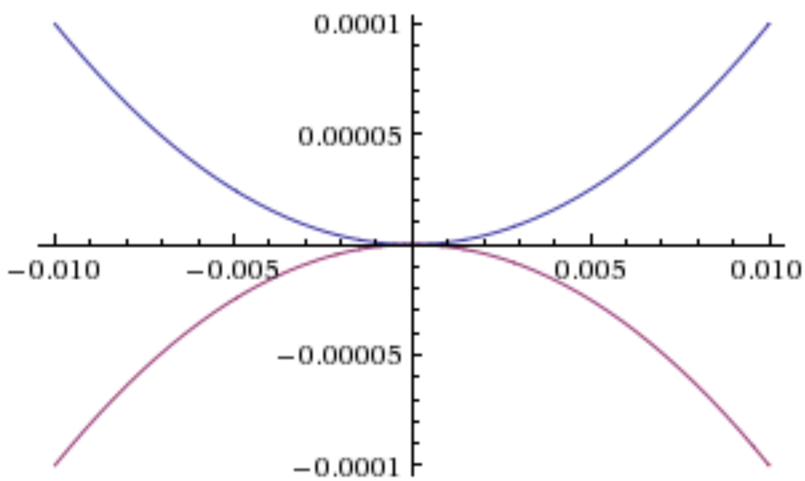
We now pivot slightly to talk about a technique for computing some difficult limits, called the squeeze theorem. The basic idea of the squeeze theorem is this: if you don't know how to evaluate the limit of a complicated function, sometimes it is sufficient to find some *simpler, continuous* functions that manage to "squeeze the complicated function between them. It's easiest to understand this in an example.

*Example* 5.1. What is $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$?

What makes this limit tricky is the term $\sin\left(\frac{1}{x}\right)$. We saw in an earlier example (see the end of the lecture 3 notes) that this function not only isn't defined at $x = 0$, it also has no limit there (it oscillates more and more wildly near 0). But if you graph this function, you will see something rather different. Below I show the graph of this function from $x = -0.01$ to $x = 0.01$.




As you can see, this function still has the more-and-more-rapid oscillation behavior of $\sin\left(\frac{1}{x}\right)$, but it nevertheless zeros in on a limit in the middle. It very much appears that the limit is 0. Indeed, you can make out from the peaks and troughs of this graph two frontiers that appear to squeeze the graph between themselves.




These two frontiers are the graphs $y = x^2$ and $y = -x^2$. The reason they "squeeze" the graph is that

$$x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$
and
$$x^2 \sin\left(\frac{1}{x}\right) \geq -x^2.$$

In turn, the reason for this is that $\sin x$ always lies between $-1$ and 1.

The result of this "squeezing" is that no matter how wildly the graph oscillates, it still gets neatly pinched in between two much simpler functions. This forces the limit as $x \to 0$ to be 0, since that is the number pinched between the two functions.

This idea (and argument) will be made more precise by the squeeze theorem.

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The squeeze theorem simply says that in situations like above, where we can squeeze a function between two other functions with the same limit in the middle, then we can use this to find its limit.

**Theorem 5.2** (Squeeze theorem)**.** *If $g, f, h$ are three functions such that $g(x) \leq f(x) \leq h(x)$ and $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} h(x) = L$ for some number $c$, then $\lim\limits_{x\to c} f(x)$ exists and is equal to $L$ also.*

*The same statement is true if all limits are replaced with limits from above, or limits from below.*

In the example above, here are our three functions:

$$\begin{aligned} g(x) &= -x^2 \\ f(x) &= x^2 \sin\left(\frac{1}{x}\right) \\ h(x) &= x^2 \end{aligned}$$

Since $g(x)$ and $h(x)$ are both *continuous*, their limits as $x \to 0$ are equal to their values, which are $g(0) = 0$ and $h(0) = 0$. These are equal, so the limit of $f(x)$ is also 0.

Here are two more examples of how this technique can be applied. The first is a slight variation of the previous example.

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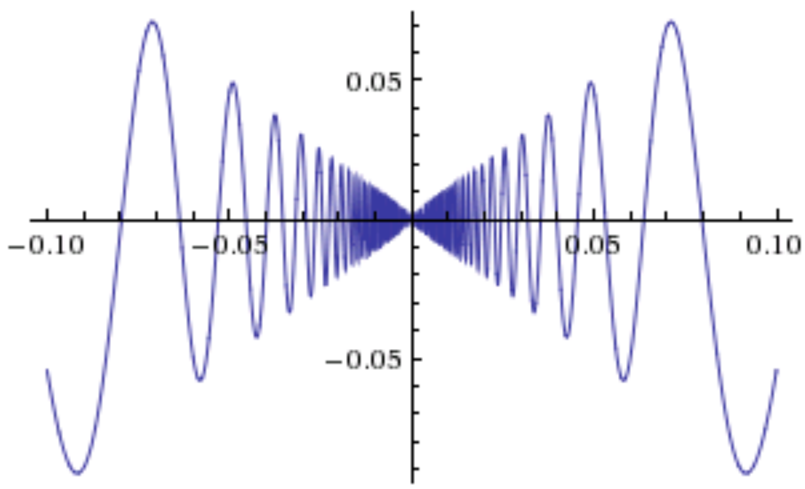
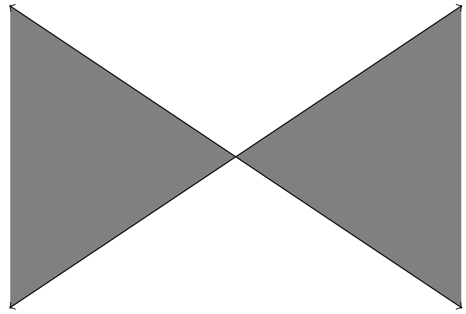
*Example* 5.3. What is $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$?

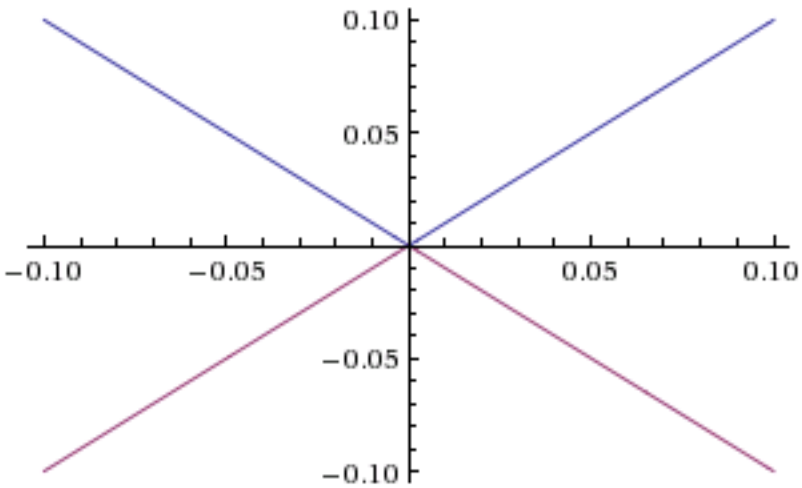
*Solution.* Here's what the graph of this function looks like. Note that I've zoomed in very far to see this behavior (see the labels on the axes).



We can see that this graph seems to be squeezed into a "bow-tie" region, like the following.



It is tempting to say that the two bounding graphs should be $y = x$ and $y = -x$ but that isn't quite right – you want to know the *upper boundary* of the bow-tie, and the lower boundary. You want the following two functions (shown in red and blue).




The bounds you actually want are $\pm|x|$. We can apply the squeeze theorem as follows.

$$\begin{array}{lcl} -|x| \leq & x\sin\left(\frac{1}{x}\right) & \leq |x| \\ \lim_{x\to 0} -|x| & = 0 = & \lim_{x\to 0} |x| \\ \text{therefore} & & \lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = 0 \text{ as well.} \end{array}$$

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*Example* 5.4. What is $\lim_{x\to 0}\left(e^x + x^{10}\sin(\ln x)\right)$?

*Solution.* The complicated part of this function is the $\sin(\ln(x))$ bit, which is very difficult to study near 0 since $\ln x$ has a vertical asymptote at $x = 0$. Fortunately, we can still leverage the fact that $|\sin x| \leq 1$ for all $x$. From this we obtain

$$-x^{10} \leq x^{10}\sin(\ln x) \leq x^{10}$$

and therefore also

$$e^x - x^{10} \leq x^{10}\sin(\ln x) \leq e^x + x^{10}$$

The functions on both sides here are continuous; and each has value $e^0 = 1$ at $x = 0$. So we can conclude, by the (one-sided) squeeze theorem, that $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 1$.

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# 6 Appendix: A precise definition of limits and continuity

We work fairly informally with limits and continuity in this course, but I will mention the usual formal definition in case any students are curious. As usual, material in this appendix has no bearing on any of the assigned homework or exams.

I will begin with continuity, which actually has a slightly simpler-to-state definition.

**Definition 6.1.** The function $f(x)$ is *continuous* at a value $c$ if

- For every positive number $\epsilon$,
- there exists some positive number $\delta$,
- such that whenever $x$ satisfies $|x - c| < \delta$, $f(x)$ satisfies $|f(x) - f(c)| < \epsilon$.

This definition is very abstract and initially opaque. But in fact, all it does is state in precise language what I've been saying about inaccurate experiments getting close to the ideal result. The positive number $\epsilon$ here is just the level of accuracy of the measure stick you use to measure the outcome of the experiment. What the definition says is that no matter how accurate your measuring stick is, your experiment will give a result that is correct (to within this level of accuracy) so long as you do the experiment with sufficient precision. The "sufficient precision" is specified by the number $\delta$, which tells how close to the desired value ($c$) you have to get with your experiment to measure the correct outcome.

I should remark that you can also, of course, first define limits (as I do below) and then define continuity as we do above ($f$ is continuous at $c$ if $\lim_{x\to c} f(x) = f(c)$).

The definition of limit takes essentially the same form. The limit is essentially the value that $f(c)$ would *have to be* in order to make the function $f$ continuous at $c$.

**Definition 6.2.** The *limit* $\lim_{x\to c} f(x)$ is the unique number $L$ such that

- For every positive number $\epsilon$,
- there exists some positive number $\delta$,
- such that whenever $x$ satisfies $0 < |x - c| < \delta$, $f(x)$ satisfies $|f(x) - L| < \epsilon$.

If there is no such number $L$, then we say that the limit does not exist.

I leave it to you to think about how to modify this definition to define limits from the left and from the right (one-sided limits), rather than two-sided limits.

There is one slightly subtle different between this definition and the first one: the condition that $0 < |x-c|$. All this says is that we only consider experiments that are not precisely correct. Even if the value $f(c)$ happens to be defined, we *do not care what it is* in computing the limit. It seems a little bizarre that we should leave out the precisely correct input when defining the "ideal limit;" you should think about why this is necessary (if you are interested in this sort of thing). One example of why it is needed is the case of a jump discontinuity, where we want to make sure that the limit still exists from both sides, even though the value of the function can be only one of them.