Lecture 33: Area by slicing

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1 Introduction

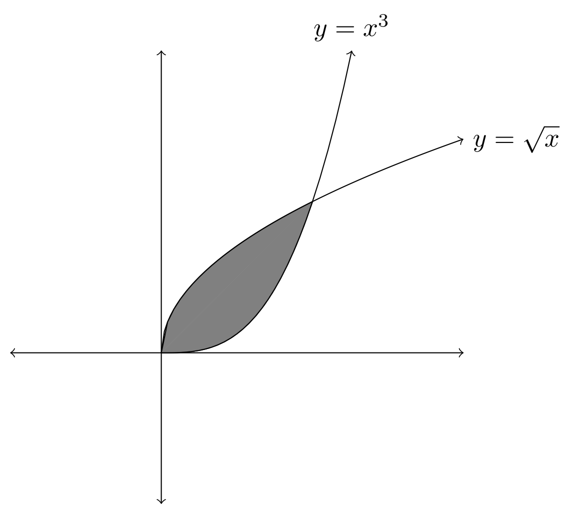
In this lecture and the next, we'll revisit the idea of Riemann sums, and show how it can be used to convert certain problems to computing integrals. Today we begin with the problem of finding areas in the plane: we will specify some curves bounding a region, and compute the area of that region by slicing it into many pieces, interpreting this as a Riemann sum, and transforming it into an integral.

On central theme in this class and the next is that *there are multiple ways to slice.* When you slice differently, you get a different-looking integral, but the answer (of course) will always be the same. This has one nice consequence: you can sometimes compute a difficult integral by re-slicing the area under a curve and computing an easier integral.

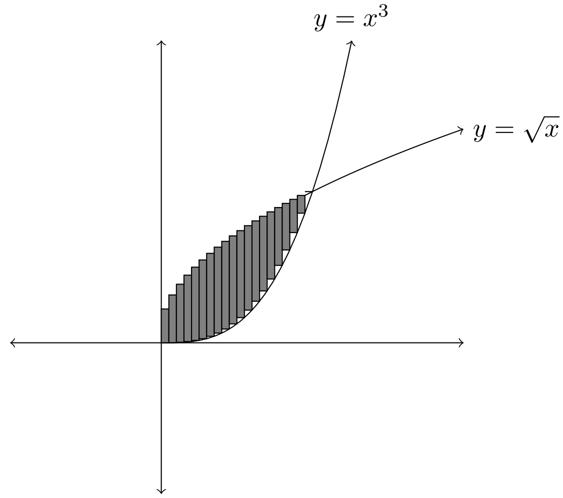
The reference for today is Stewart §6.1.

2 Slicing: vertical and horizontal

Consider the region bounded by the curves $y = \sqrt{x}$ and $y = x^3$. This region is shown shaded below.



We can imagine computing this area just like we computed the area under curves when defining the definite integral: first cut the region up into n vertical pieces (where n is some number that's not too small), and approximate each of these pieces with a rectangle. To get the top and bottom bounds of the rectangle, I'll use the curves at the right side of the slice; so this is a right-hand approximation.



We can give names to the x-values at the ends of the rectangles: say that the first rectangle goes from $x = x_0$ to $x = x_1$, the second goes from $x = x_1$ to $x = x_2$, and so forth. As usual, denote the difference between consecutive x_k values as Δx. These values run from $x_0 = 0$ to $x_n = 1$, so we can write them precisely as follows.

$$\begin{aligned} \Delta x &= 1/n \\ x_k &= k \cdot \Delta x \end{aligned}$$

Now focus on the kth rectangle in the slicing above. It is bounded by $x = x_{k-1}$ on the left, by $x = x_k$ on the right, by $y = x_k^3$ on the bottom and $y = \sqrt{x_k}$ on the top. Therefore it has width Δx and height $(\sqrt{x_k} - x_k^3)$, therefore area $(\sqrt{x_k} - x_k^3) \cdot \Delta x$.

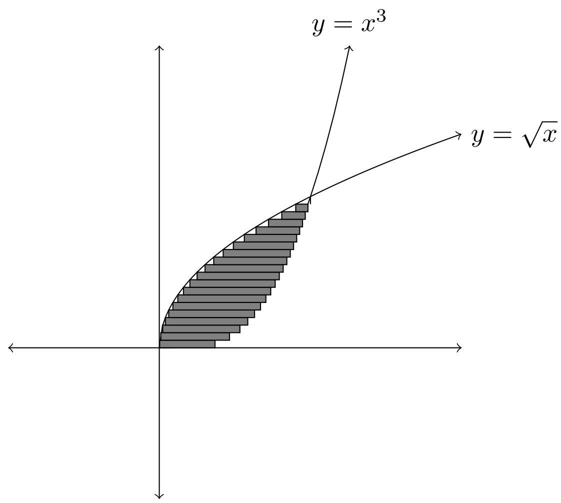
Adding all this areas together gives an estimate for the area of the region bounded by $y = \sqrt{x}$ and $y = x^3$.

$$\text{Area} \approx \sum_{k=1}^{n} (\sqrt{x_k} - x_k^3) \cdot \Delta x$$

Now, I can recognize this area as a Riemann sum. The exact area will be the limit as $n \to \infty$, which will be any integral of the form $\int_{\cdot}^{\cdot} (\sqrt{x} - x^3) dx$ (the x_k becomes an x, and the Δx becomes a dx, when transforming from a sum to an integral). To get the limits of integration, just see what the smallest and largest x values are: the smallest is $x_0 = 0$, and the largest is $x_n = 1$. So the area is the following integral.

$$\begin{aligned} \text{Area} &= \int_0^1 \left(\sqrt{x} - x^3\right) dx \\ &= \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^4\right]_0^1 \\ &= (\frac{2}{3} - \frac{1}{4}) - (0 - 0) \\ &= \frac{5}{12} \end{aligned}$$

Now, there is a second way to compute this same area: we could instead slice the region into horizontal pieces, as shown.



In order to turn this slicing method into an integral, we must switch the roles of x and y. Now the height of the kth rectangle is Δy, while the width will be the different in x values for a particular values of y. To get this width, we must convert the functions of y to functions of x.

$$\begin{aligned} y = x^3 \quad &\Leftrightarrow \quad x = \sqrt[3]{y} \\ y = \sqrt{x} \quad &\Leftrightarrow \quad x = y^2 \end{aligned}$$

Now the right boundary of the kth slice is given by $\sqrt[3]{y_k}$, while the left boundary is given by y_k^2. So the Riemann sum approximation is

$$\sum_{k=1}^{n} (\sqrt[3]{y} - y^2) \cdot \Delta y \qquad \text{where } \Delta y = \frac{1}{n},\ y_k = k \cdot \Delta y.$$

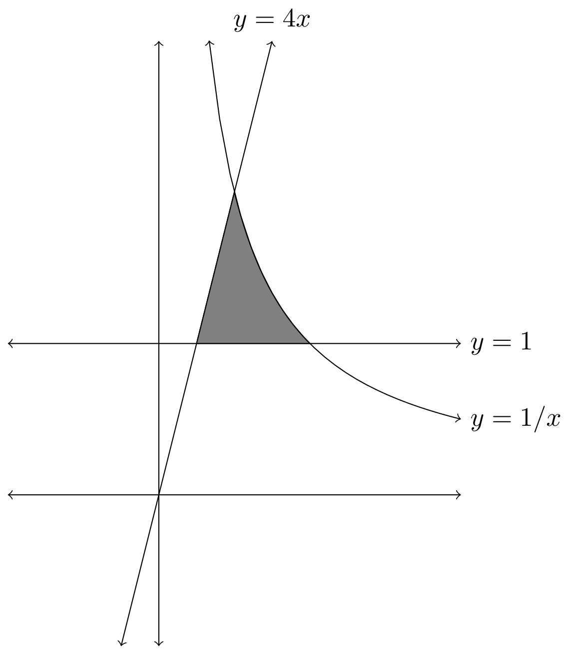
This becomes the following integral.

$$\begin{aligned} \int_0^1 (\sqrt[3]{y} - y^2) dy &= \left[\frac{3}{4} y^{4/3} - \frac{1}{3} y^3 \right]_0^1 \\ &= (\frac{3}{4} - \frac{1}{3}) - (0 - 0) \\ &= \frac{5}{12} \end{aligned}$$

Note that I chose the limits of integration here because 0 is the smallest value of y in the region, while 1 is the largest value of y in the region.

3 Choosing a slicing method

The example above was a situation where it was tractable to slice either horizontally or vertically. It some cases, however, one slicing method is somewhat easier. For example, consider the region bounded by $y = 1, y = 4x$, and $y = 1/x$, shown below.



Again, we could attempt to slice this area either vertically or horizontally. In this case, however, horizontal slicing will be somewhat easier. The reason is that the left boundary is always the same curve: $y = 4x$, or $x = \frac{1}{4}y$, and the right boundary is also always the same curve: $y = 1/x$, or $x = 1/y$. So we can express this area as a one integral. Here, I'll skip all the Riemann sums and go straight to the integral. Consult the previous section to see how to come up with an expression like this.

$$
\begin{aligned}
\text{Area} &= \int_{(\text{smallest } y)}^{(\text{largest } y)} ((\text{rightmost } x\text{-coordinate}) - (\text{leftmost } x\text{-coordinate}))\, dy \\
&= \int_1^2 \left(\frac{1}{y} - \frac{1}{4}y \right) dy \\
&= \left[\ln|y| - \frac{1}{8}y^2 \right]_1^2 \\
&= (\ln 2 - \frac{1}{8} \cdot 4) - (\ln 1 - \frac{1}{8} \cdot 1) \\
&= \ln 2 - \frac{1}{2} + \frac{1}{8} \\
&= \ln 2 - \frac{3}{8} \ (\approx 0.318)
\end{aligned}
$$

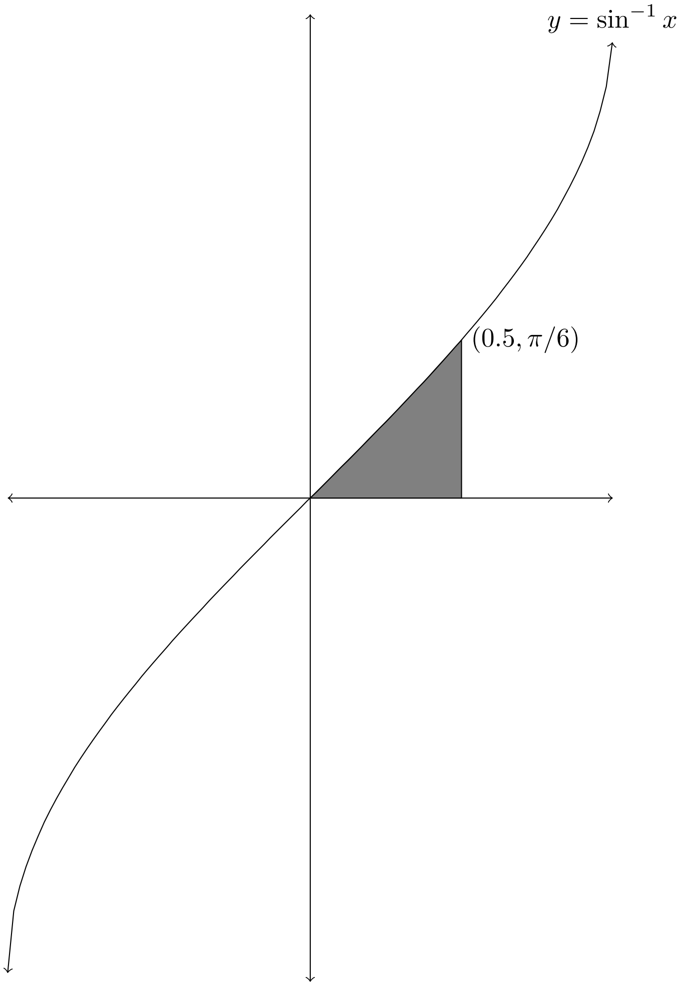
We could also slice vertically. However, **slicing vertically is more difficult** because the top bounding curve changes partway through (when we slice horizontally it is the same across the whole region). It can still be done, it just involves splitting the area into two integrals: the result would be $\int_{\frac{1}{4}}^{\frac{1}{2}} (4x-1)dx + \int_{\frac{1}{2}}^{1} (\frac{1}{x}-1)dx$. Of course this computation would come out to the same thing, but it takes a bit more work.

4 Re-slicing definite integrals

In some cases, changing the direction of slicing is useful to evaluate definite integrals. For example, consider the following integral.

$$\int_0^{1/2} \sin^{-1} x dx$$

It turns out that you can evaluate this integral using integration by parts. However, there is a somewhat easier way, if you are willing to change your point of view slightly. Remember that you can think of this integral as the signed area shown.



The integral $\int_0^{1/2} \sin^{-1} dx$ slices this area vertically. But there's no reason we must do this: we could also slice horizontally. Then this is the area between $x = \sin y$ on the left and $x = 1/2$ on the right. So we can re-express this area as an integral in y, which (happily) is much easier to compute.

All we need to know are the left and right boundaries, in terms of y (these are $\sin y$ and $\frac{1}{2}$, respectively), and the largest and smallest values of y (these are $\pi/6$ and 0, respectively). This gives the following integral.

$$\begin{aligned}
\int_0^{1/2} \sin^{-1} x dx &= \int_0^{\pi/6} (\frac{1}{2} - \sin y) dy \\
&= \left[\frac{1}{2} y + \cos y \right]_0^{\pi/6} \\
&= (\frac{\pi}{12} + \frac{\sqrt{3}}{2}) - (0+1) \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \quad (\approx 0.128)
\end{aligned}$$

By the way, you may notice in the picture that the shape of this region is extremely close to being a perfect triangle. Indeed, the area you would get if you just assumed it to be a triangle is within about 0.003 of the exact area.

Aside. Actually, this idea of "re-slicing" is really a special case of integration by parts; any integral you can do by replacing can also be done by integration by parts (specifically, by taking $u = f(x), dv = dx$), but it's not as clear what's going on in the algebra. See the appendix to lecture 32 for some discussion of this.