Lecture 29: Evaluating definite integrals

Nathan Pflueger

18 November 2013

1 Introduction

The fundamental theorem of calculus provides the basic tool for evaluating definite integrals: if you can find an antiderivative of a function, then you can compute any definite integrals of it. In this lecture, we explain how this words and give some examples.

The reference for today is Stewart §5.3.

2 Evaluating integrals

Consider an example first: let $f(x) = x$. We saw before (using geometry) that this has area function $\int_0^x t dt$ given by $F(x) = \frac{1}{2}x^2$. It turns out that if we want any definite integral of $f(t)$, we can get it using this one area function. For example:

$$\begin{aligned} \int_3^5 t dt &= \int_0^5 t dt - \int_0^3 t dt \\ &= F(5) - F(3) \\ &= (\frac{1}{2}5^2) - (\frac{1}{2}3^2) \\ &= \frac{25}{2} - \frac{9}{2} \\ &= 8 \end{aligned}$$

The reason that this computation worked is because of the basic arithmetic of definite integrals, and in particular that $\int_3^5 = \int_0^5 - \int_0^3$.

Now, $F(x) = \int_0^x t dt$ was only one particular choice of an area function. We also could have chosen $\int_{-1}^x t dt$ instead, or $\int_{-7}^x t dt$. But it actually *doesn't matter which one we chose*: because for example $\int_3^5 = \int_{-7}^5 - \int_{-7}^3$ as well. When we evaluate definite integrals, it doesn't matter which area function we use.

Now recall the fundamental theorem of calculus (one version of it): *any* area function $F(x) = \int_c^x f(t)dt$ (where c is a constant) is *an antiderivative* of $f(x)$: that is, $F'(x) = f(x)$. Choosing a different area function (like $\int_{-7}^x t dt$ in the example above) just amounts to choosing a different antiderivative.

The upshot of all of this is the follow "evaluating theorem," which you should think of as another version of the fundamental theorem of calculus.

$$\begin{aligned} \int_a^b f(x)dx &= F(b) - F(a) \\ \text{where} \quad & F(x) \text{ is any antiderivative of } f(x). \end{aligned}$$

To see why *any* antiderivative will work: notice that you could choose a different one (like $F(x) + 7$), which will differ form the first by a constant, but then when you compute $F(b) - F(a)$ that constant will cancel (e.g. $(F(b)+7)-(F(a)+7)=F(b)-F(a)$). So it doesn't matter which antiderivative you pick; you should reach for the most convenient one.

You might also see this fact written in the following way. This is the same statement; here f just plays the role of F and f' plays the role of f.

$$\int_a^b f'(x)dx = f(b) - f(a)$$

This version of the statement might feel more obvious to you (it certainly does to be): it says that the difference in a quantity between two times (a and b) is given by adding up (accumulating) the derivative of the function from one time to the other.

3 Summary and shorthand

To summarize the discussion above, here's the standard notation for computing definite integrals. Here as usual $F(x)$ means your favorite antiderivative of $f(x)$.

$$\begin{aligned}\int_a^b f(x)dx &= [F(x)]_a^b \\ &= F(b) - F(a)\end{aligned}$$

The notation $[F(x)]_a^b$ is just a bit of shorthand, and it means exactly what's written in the second line: evaluate $F(x)$ and b and a, and subtract the latter from the former.

For example:

$$\begin{aligned}\int_3^7 2x\ dx &= \left[x^2\right]_3^7 \\ &= 7^2 - 3^2 \\ &= 40\end{aligned}$$

Here, the very first equals sign is justified because the derivative of x^2 is $2x$. It doesn't matter *how* I come up with x^2; all that matters is that its derivative is $2x$ (which I can easily check). Evaluating integrals is often about inspired guessing. Fortunately, there are also some good standard methods, which we'll start to study next time.

4 Summary of key antiderivatives

As you can see, the hard part of evaluating definite integrals algebraically is usually finding an antiderivative. As we've said before, this is fiendishly difficult in general (a classic example is e^{x^2}, which is actually known to be impossible to antidifferentiate using simple functions). The main methods all revolve on re-expressing a simplifying the function you hope to integrate, with the goal of eventually reducing the problem to integrating a simple function whose antiderivative is known.

For this reason, it is crucial that you know the most common antiderivatives very well. Not only will you need to recall them often, but you'll need to *look for them,* so that you can detect cases where a complicated integral can be reduced to a simple one. Therefore we suggest that you learn, at a minimum, the following antiderivatives, well enough that you don't have to think at all to produce them.

$f(x)$	$F(x)$
x^{-1}	$\ln\|x\|$
$1/\sqrt{x}$	$2\sqrt{x}$
1	x
$\sqrt{x}$	$\frac{2}{3}x^{3/2}$
x	$\frac{1}{2}x^2$
x^n(any constant $n \neq 1$)	$\frac{1}{n+1}x^{n+1}$
e^x	e^x
a^x(any constant $a > 0$)	$\frac{1}{\ln a}a^x$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \ \tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$

Of these, the hardest to remember are the last two. You may just want to memorize them. One clue that helps me: expression like $\sqrt{1-x^2}$ and $1+x^2$ are dead giveaways that the Pythagorean theorem is lurking somewhere nearby. And indeed both of these derivatives were obtained by using the Pythagorean theorem to work with trig and inverse trig functions. So when you see these, this may remind you that there is trigonometry at work.

In any case, just know these antiderivatives. You've probably noticed that I hate memorization, so I favor *rehearsing* over memorizing (practicing the derivations until it becomes instant), but do whatever works for you.

5 Examples

Here are some examples of definite integrals. Computations of each one follow, along with some commentary.

1. $$\int_1^4 3\sqrt{x}dx$$

2. $$\int_0^\pi 5\sin\theta d\theta$$

3. $$\int_x^{7x} \frac{dt}{t}$$

4.

$$\int_0^{10} e^{-t} dt$$

5.

$$\int_h^1 \frac{1}{\sqrt{x}} dx$$

6.

$$\int_{-\pi/3}^{\pi/3} \left(2 - \sec^2 x\right) dx$$

7.

$$\int_{-1}^1 \frac{1}{1+x^2} dx$$

8.

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} dx$$

Here are computations for each of these integrals, with some commentary in some cases.

1.

$$\begin{aligned}\int_1^4 3\sqrt{x} dx &= \left[2x^{3/2}\right]_1^4 \\ &= 2 \cdot 4^{3/2} - 2 \cdot 1^{3/2} \\ &= 2 \cdot 8 - 2 \\ &= 14\end{aligned}$$

Here, we know that an antiderivative of $\sqrt{x}$ is $\frac{2}{3}x^{3/2}$. To get an antiderivative of $3\sqrt{x}$, I just multiplied by 3.

2.

$$\begin{aligned}\int_0^{\pi} 5 \sin\theta d\theta &= [5(-\cos\theta)]_0^{\pi} \\ &= 5 \cdot (-(-1)) - 5 \cdot (-1) \\ &= 5 + 5 \\ &= 10\end{aligned}$$

3.

$$\begin{aligned}\int_x^{7x} \frac{dt}{t} &= [\ln|t|]_x^{7x} \\ &= \ln|7x| - \ln|x| \\ &= \ln\left|\frac{7x}{x}\right| \\ &= \ln 7\end{aligned}$$

The variable x here turns out not to matter. At the end of the computation, this integral does not depend on x.

4.

$$\begin{aligned}\int_0^{10} e^{-t} dt &= \left[-e^{-t}\right]_0^{10} \\ &= -e^{-10} - (-e^0) \\ &= -e^{-10} + 1\end{aligned}$$

For this problem, we could not find the antiderivative of e^{-t} immediately by looking at the table of common antiderivatives. However, you can observe that since the derivative of e^{-t} is $-e^{-t}$ (by an easy application of the chain rule), the derivative of $-e^{-t}$ is e^{-t}, so we can find the antiderivative by a little guesswork in this case.

5.

$$\begin{aligned}\int_h^1 \frac{1}{\sqrt{x}} dx &= \left[2\sqrt{x}\right]_h^1 \\ &= 2\sqrt{1} - 2\sqrt{h} \\ &= 2 - 2\sqrt{h}\end{aligned}$$

6.

$$\begin{aligned}\int_{-\pi/3}^{\pi/3} \left(2 - \sec^2 x\right) dx &= [2x - \tan x]_{-\pi/3}^{\pi/3} \\ &= \left[2 \cdot \frac{\pi}{3} - \sqrt{3}\right] - \left[2 \cdot (-\frac{\pi}{3}) - (-\sqrt{3})\right] \\ &= \frac{4}{3}\pi - 2\sqrt{3}\end{aligned}$$

In this case, we knew the antiderivatives of 2 ($2x$) and of $\sec^2 x$ ($\tan x$). Since the derivative of a different is the difference of the derivatives, we know can get an antiderivative of $2 - \sec^2 x$ by subtracting $\tan x$ from $2x$.

7.

$$\begin{aligned}\int_{-1}^{1} \frac{1}{1+x^2} dx &= \left[\tan^{-1} x\right]_{-1}^{1} \\ &= \tan^{-1} 1 - \tan^{-1}(-1) \\ &= \frac{\pi}{4} - (-\frac{\pi}{4}) \\ &= \frac{\pi}{2}\end{aligned}$$

8.

$$\begin{aligned}\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} dx &= \left[\sin^{-1} x\right]_{-1/2}^{1/2} \\ &= \sin^{-1}(1/2) - \sin^{-1}(-1/2) \\ &= \frac{\pi}{6} - (-\frac{\pi}{6}) \\ &= \frac{\pi}{3}\end{aligned}$$