Lecture 28: The fundamental theorem of calculus

Nathan Pflueger

15 November 2013

1 Introduction

Today we discuss the basic theorem about integration. It says that integration and differentiation are essentially inverse processes – and that in particular integrals can be evaluated by finding antiderivatives. Next week we'll turn our attention to how this theorem is used to do computations.

The reference for today is Stewart §5.4.

2 Area functions

If $f(x)$ is any function, and c is any constant value, we can define a new function, called an *area function,* as follows.

$$F(x) = \int_c^x f(t)dt$$

This is called an area function because it tells the area under the curve $y = f(x)$ for *many* different values of x (so it is a function, not a single area).

Example 2.1. Consider the function $f(x) = x$, and take $c = 0$. Then we can define the following area function.

$$F(x) = \int_0^x tdt$$

Note that I have changed the variable x to a t in the integrand (the stuff after the $\int$ sign), because I am already using the letter x for a different purpose in the endpoints of the integral.

The function $F(x)$ is easy to find explicitly: if $x > 0$ it just gives the area of a right triangle of legs x and x, which is $\frac{1}{2}x^2$. So for $x > 0$, $F(x) = \frac{1}{2}x^2$. On the other hand, if $x < 0$, then the interval $[0, x]$ is backwards, so we can rewrite $F(x) = -\int_x^0 tdt$. In turn, $\int_x^0 tdt$ is the *signed* area of a triangle lying *below* the x axis, both of whose legs have length $(-x)$. So we can conclude that for $x < 0$, $\int_0^x tdt = -(-\frac{1}{2}(-x)^2) = \frac{1}{2}x^2$. So despite all the minus signs that float around when we think about signed area and backwards intervals, it turns out that $F(x) = \frac{1}{2}x^2$ for *all* values of x.

Example 2.2. (Net profit). Suppose that $p(t)$ tells the rate at which a business is making profit, expressed (say) in dollars per day. Then the area function $P(T) = \int_7^T p(t)dt$ simply gives the *net profit* that the company makes between day 7 and day T.

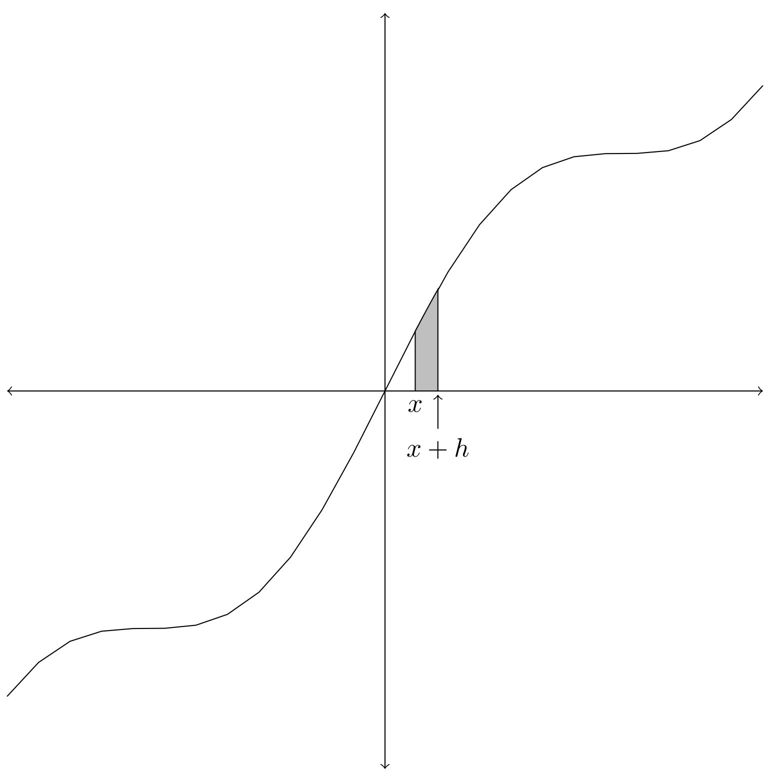
This second example shows one of the principle ways that area functions arise in practice: they give the *total accumulation* of some quantity over time, given the rate of accumulation at various times.

3 The derivative of an area function

Area functions provide the fundamental bridge between the two main branches of calculus (integration and differentiation). The reason for this bridge is that area functions have very interesting derivatives. To see why, think about what exactly the definition of the derivative is. To simplify it, we will use some of the main properties of definite integrals discussed in the previous class.

$$\begin{aligned} F(x) &= \int_c^x f(t)dt \\ F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_c^{x+h} f(t)dt - \int_c^x f(t)dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \end{aligned}$$

Now, what exactly is this new integral, $\int_x^{x+h} f(t)dt$? It is essentially the area of a little *slice* underneath the graph of $y = f(t)$.



So we're studying the area of a tiny little slice of width h, and then *dividing by* the width h of the slice.

The basic insight here is that one way to look at this quantity $\frac{1}{h}\int_x^{x+h} f(t)dt$ is that it is essentially expressing the *average value* of $f(t)$ on the very narrow interval $[x, x+h]$. Since we are taking a limit as $h \to 0$, we are interesting in taking this average on smaller and smaller intervals. When this window gets very small – that is, as $h \to 0$ in the limit – the point is that as long as $f(t)$ is a continuous function, *it doesn't vary much* in $[x, x+h]$. So the average will be basically the same thing as $f(x)$ itself. So we can guess, in fact, that:

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \\ &= f(x) \end{aligned}$$

This guess is in fact precisely correct if $f(x)$ is continuous; this is the fundamental theorem of calculus.

4 The fundamental theorem

As remarked at the end of the previous section, we have the following theorem.

Theorem 4.1 (Fundamental theorem of calculus)**.** *Suppose that $f(x)$ is a continuous function, and $F(x) = \int_c^x f(t)dt$, where c is some constant. Then $F'(x) = f(x)$.*

Another way to say this theorem is: *area functions are antiderivatives.* We can also write the equation of this theorem in the following way.

$$\frac{d}{dx}\int_c^x f(t)dt = f(x)$$

This expression makes plain the inverse relationship between integration and differentiation.

Note. The constant c used in the definition of the area function is *completely irrelevant to the derivative.* Different values of c will of course give different area function $F(x)$, but the points is that they all have the same derivative, namely $f(x)$.

4.1 The proof, in an illustrative special case

You can see all the moving parts of the proof of the fundamental theorem in the following special case: let's assume that $f(x)$ is **positive and increasing.**

Remember that $F(x)$ is the area function $F(x) = \int_c^x f(t)dt$, and therefore we can compute the derivative as $F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt$.

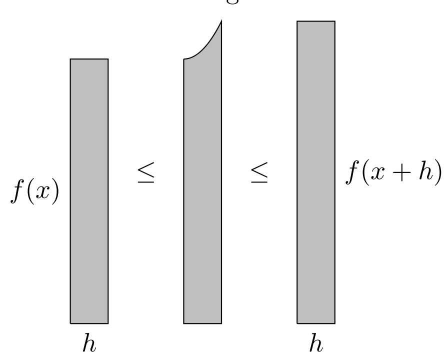
Let's use the fact that f is increasing: that means in particular that for all t in $[x, x+h]$ (the only values we're using here),

$$f(x) \leq f(t) \leq f(x+h)$$

This implies in particular that (fixing values of x and h)

$$h \cdot f(x) \leq \int_x^{x+h} f(t)dt \leq h \cdot f(x+h)$$

These two inequalities are illustrated in the picture below: the idea is that the small "slice" whose area the integral calculates is bounded between two rectangles whose areas we can write down exactly.



Now, divide all parts by h to obtain:

$$f(x) \leq \frac{1}{h}\int_x^{x+h} f(t)dt \leq f(x+h)$$

Now, these inequalities hold for all x and h. So we can take the limits of the two parts on the outside and use the squeeze theorem.

$$\begin{aligned}
\lim_{h\to 0} f(x) &= f(x) \quad \text{(doesn't depend on } h\text{)} \\
\lim_{h\to 0} f(x+h) &= f(x) \quad \text{(because } f \text{ is continuous)}
\end{aligned}$$

So it follows from the squeeze theorem that

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt &= f(x) \\
\text{i.e. } F'(x) &= f(x) \\
\text{i.e. } \frac{d}{dx}\int_c^x f(t)dt &= f(x)
\end{aligned}$$

4.2 Some examples

Here are some examples of the fundamental theorem.

1. Consider $\frac{d}{dx}\int_1^x \frac{1}{t}dt$. This is a derivative of an area function, so it is just the original function (the one being integrated):

   $$\frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}$$

   Here is a point of possible confusion: in the integrand, I've written $\frac{1}{t}$, but on the right side I've written $\frac{1}{x}$. Just remember that if you differentiate with respect to x, the result should be a function of x.

2. Consider $\frac{d}{dx}\int_{12}^x \sqrt{s^2+17}ds$. This is a derivative of an area function, so it's just the function being integrated (with the dummy variable s replaced by x).

   $$\frac{d}{dx}\int_{12}^x \sqrt{s^2+17}ds = \sqrt{x^2+17}$$

   Note that the endpoint 12 is a complete red herring here: it doesn't affect the answer at all. Area functions have the same derivative no matter which left endpoint is chosen.

3. Consider $\frac{d}{dx}\int_7^{x^2} \sin t dt$. Here we can't quite apply the fundamental theorem directly since $\int_7^{x^2} \sin t dt$ isn't in the usual form for an area function. However, we can differentiate it using the chain rule: we know that $\frac{d}{dy}\int_7^y \sin t dt = \sin y$, so taking $y = x^2$ and using the chain rule, we can write

   $$\begin{aligned}
   \frac{d}{dx}\int_7^{x^2} \sin t dt &= \sin(x^2)\cdot\frac{d}{dx}x^2 \\
   &= \sin(x^2)\cdot 2x
   \end{aligned}$$

4. Consider $\dfrac{d}{dx}\displaystyle\int_x^{100} e^t dt$. We can evaluate this be first writing $\int_x^{100} e^t dt = -\int_{100}^x e^t dt$ and then differentiating (with the fundamental theorem) to obtain

$$\frac{d}{dx}\int_x^{100} e^t dt = -e^x$$

5 Different area functions are different antiderivatives

This is just a final remark: the fundamental theorem asserts that any area function $F(x) = \displaystyle\int_c^x f(t)dt$ is an antiderivative of $f(x)$. However, they are not all the *same* antiderivative: they differ by constants. Indeed, it is very easy to find the constants, because:

$$\int_c^x f(t)dt - \int_d^x f(t)dt = \int_c^d f(t)dt$$

and if c, d are constants, than the integral $\int_c^d f(t)dt$ is also a constant. So indeed, if $F_1(x)$ and $F_2(x)$ are two different area functions for $f(x)$, then their difference will be constant, and given by a certain definite integral.