Lecture 26: Area and Riemann sums

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11 November 2013

What was least satisfying to me in our [high school] math books was the absence of any serious definition of the notion of length (of a curve), of area (of a surface), of volume (of a solid). I promised myself I would fill this gap when I had the chance. Alexandre Grothendieck, *Récoltes et Semailles*

1 Introduction

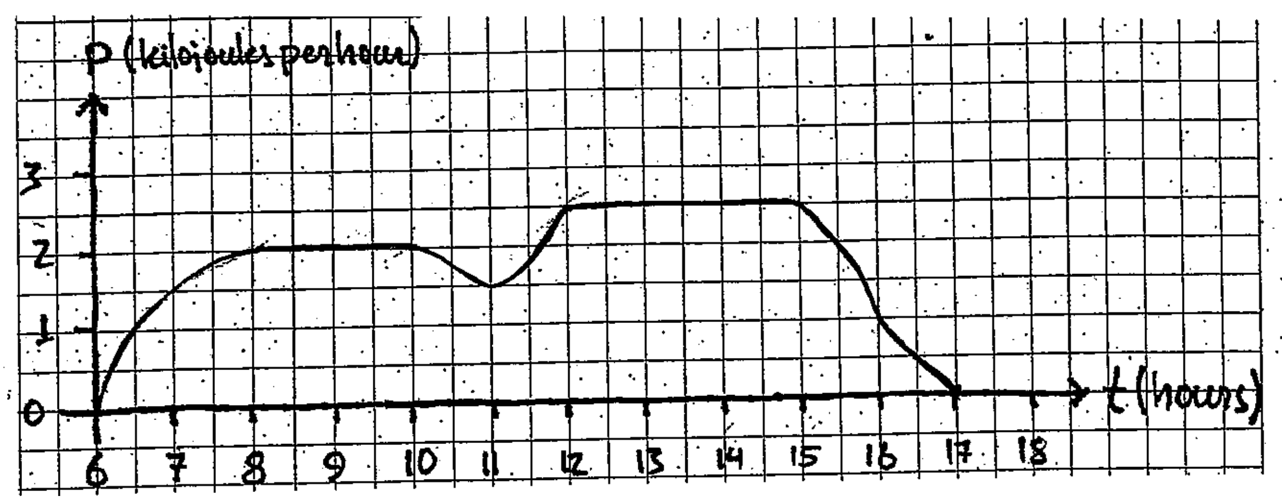
Today we consider the notion of *accumulation*. Given a function $f(x)$, which can be regarded as the rate that something is being created (it could be energy, income, population, or many other things), we can attempt to measure the total accumulation over some period of time. This problem has a very simple geometric interpretation: it corresponds to measuring the area of the region under the graph of the curve $y = f(x)$.

There is a standard mechanical way to approximate the area under a curve, frequently called a Riemann sum. In this lecture, we will introduce the problem of calculating area under a curve with a few examples, and then discuss the method of Riemann sums. Riemann sums provide one convenient way to define "area" in a precise way, which we will discuss next time.

The reference for today is Stewart §5.1.

2 The area problem

Suppose that you are generating energy with a solar panel. Naturally, its output depends on the time of day, and the current weather conditions. Suppose that the following graph shows its output (measured in kilojoules per hour) over the course of a day. Here, the variable t is the number of hours since midnight (so $t = 6$ is 6am, and $t = 15$ is 3pm, for example).



Consider the following questions:

1. How much energy (in kilojoules) does the panel produce between 12pm and 3pm?
2. How much energy (in kilojoules) does the panel produce between 8am and 10am?
3. How much energy (in kilojoules) does the panel produce between 10am 12pm?
4. How much energy (in kilojoules) does the panel produce between 6:30am and 4:30pm?

First consider the period from 12pm to 3pm. Looking at the chart, we see that the output of the panel is nearly constant during that window – it is producing 2.5 kJ per hour. The period is 3 hours long, so to compute the total energy produced, just multiply: 2.5 kJ $\cdot$3hr = 7.5kJ.

The period from 8am to 10am is similar – here the power is roughly constant at 2kJ per hour, so the total energy produced is $2 \cdot 2 = 4$ kJ.

The period from 10am to 12pm is slightly trickier – there is a dip in output in the middle (perhaps due to increased cloud coverage during that time), resulting in output as low as 1.5 kJ/hr, but as high as 2.5 kJ/hr. How can we go about finding the total energy?

The basic insight is that **the total energy is the area under the curve.** When we discuss Riemann sums in the next section, it will become more clear why this is the right way to think of total accumulated energy. In the first two cases (12pm to 3pm and 8am to 10am), this area was just given by a rectangle. For the region between 10am and 12pm, however, it is not such a simple shape. However, if all we want is a rough estimate, we can just count the grid squares under the curve (each grid square has area 1/4). There are 12 grid squares contained entirely below the curve, two more that are almost completely contained, and then small shards of three more squares (each occupying about a quarter of the square). So there are roughly 15 grid squares total under the curve (remember, this is only a very rough estimate), for an area of $15/4 = 3.75$. So we can say that roughly 3.75 kJ of energy were produced between 10am and 12pm.

We'll leave the fourth question for the next section. The answer is somewhere between 19 and 20 kJ, and there are a number of ways to go about estimating it.

Question: if you where more patient, how would you measure the area under this curve more exactly? This is, in some sense, getting at the heart of the issue in the quotation at the beginning of this lecture – how do we define and measure area anyway?

One good option would be to start subdividing the grid squares in this picture (say, dividing each one into 4 or 16 smaller squares) before counting the grid squares under the curve. Another way is encapsulated by the idea of Riemann sums, which we now discuss.

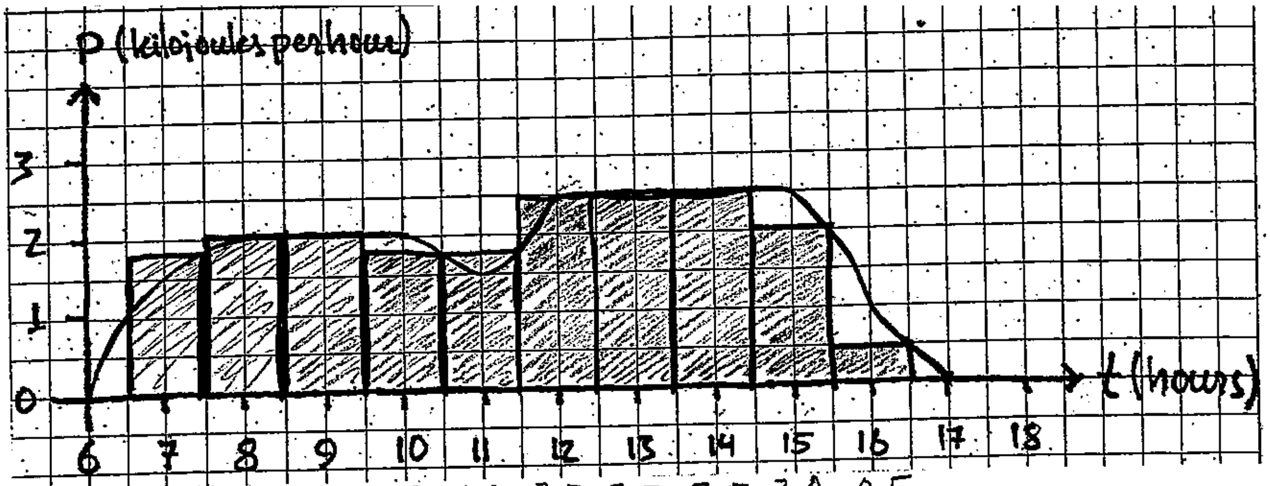
3 Riemann sums, less formally

Let's turn to the fourth problem mentioned above (the energy produced from 6:30am to 4:30pm), and go about it in a more systematic way.

A very common method for systematically estimating areas under curves is to approximate the area with the sum of the areas of many rectangles. The resulting estimate is called a *Riemann sum*, after Bernhard Riemann (1826-1866), who first used them systematically to study areas under curves. In this special case, the method is the following.

- Split the interval (6.5 to 16.5) into 10 equal subintervals (6.5 to 7.5, then 7.5 to 8.5, and so forth).
- In each subinterval, draw a rectangle, whose height is given by the power output of the panel at the *right* endpoint of the interval.
- Compute the areas of these rectangles and add them together. The result is an approximation of the desired area under the curve.

Here is a picture of what happens when you apply this method to the problem at hand.



$R_{10} = 1.8 + 2.0 + 2.0 + 1.8 + 1.8 + 2.5 + 2.5 + 2.5 + 2.0 + 0.5$

$= 19.4$

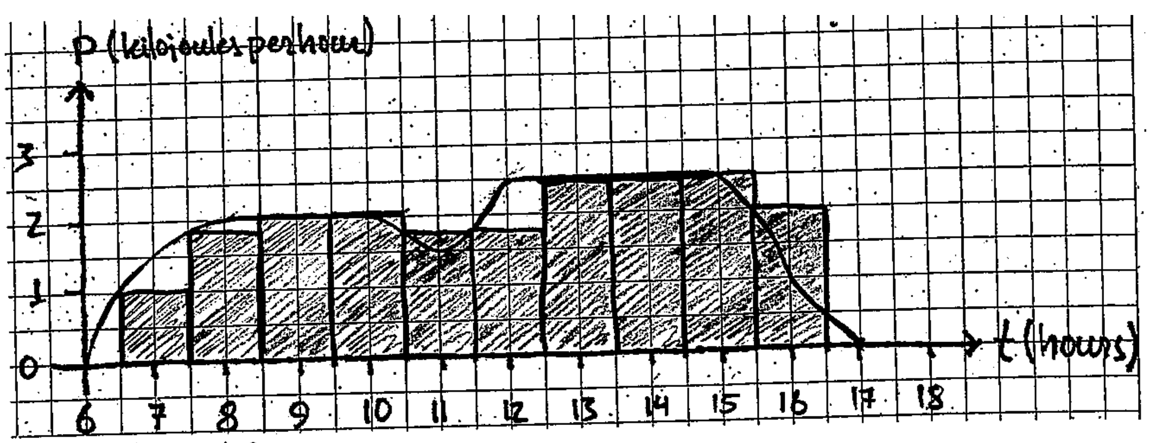
(all rectangles are width 1)

The first rectangle extends from $t = 6.5$ to $t = 7.5$. Its height is given by the power at $t = 7.5$, which is roughly 1.8 (by eyeballing the picture), so the area of this rectange is 1.8. The next rectangle extends from $t = 7.5$ to $t = 8.5$, and its heigh is given by the power at $t = 8.5$, which is 2. Proceeding in this way, we obtain 10 rectangles of various areas, and these areas add to 19.4. So this is one estimate of the area under the curve.

I have written this sum as R_{10}. The R stands for "right hand" (since the heights of all rectangles are given by the power at the right endpoint), and the 10 refers to the fact that I have split the interval $[6.5, 16.5]$ into ten equal parts.

Notice that this approximation is not perfect: where the graph of P is increasing, the rectangles are too big; where it is decreasing, the rectangles are too small. Think about why. This is a general feature of so-called right-hand Riemann sums.

Now, I could have equally well taken the left endpoint of each rectangle as my choice for the height. The result would be the following.



$L_{10} = 1.0 + 1.8 + 2.0 + 2.0 + 1.8 + 1.8 + 2.5 + 2.5 + 2.5 + 2.0$

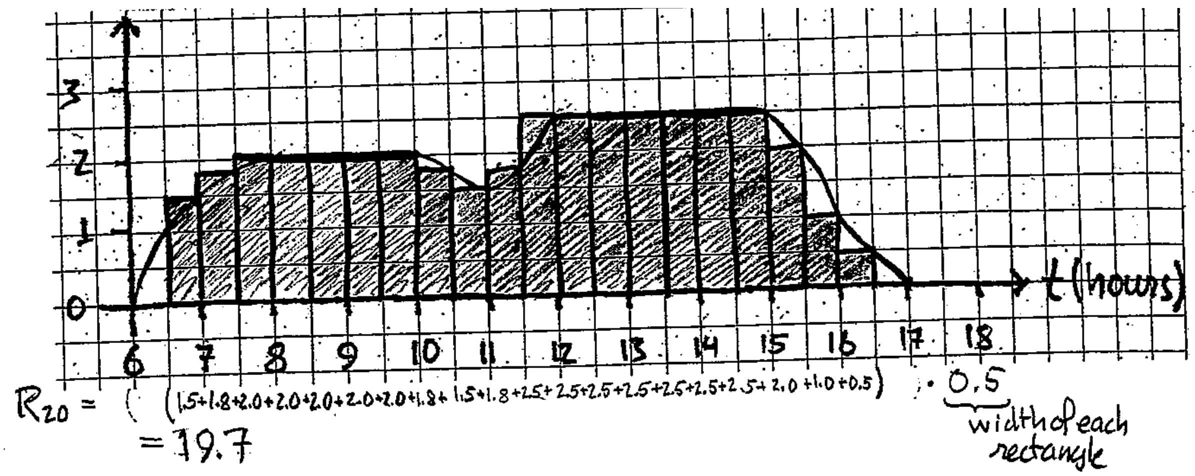
$= 19.9$

Notice that now the rectangles are too small where P is increasing, and too large where P is decreasing. Think about why.

This approximation is called L_{10}: L for left, and 10 because I am using 10 rectangles of equal width.

So far we've obtained two estimates for the desired area: 19.4 and 19.9. These are close enough that we can make a pretty educated guess that the answer is somewhere between 19 and 20.

If we want to be more accurate, we can use more rectangles. For example, here is a computation of R_20.



As you can see, this new approximation better captures the subtleties of the curve (especially around $t = 11$, but it still isn't perfect. Nonetheless, we could always take more rectangles, and make it a little bit better.

This basic method (Riemann sums) has the benefit of being quite mechanical and easy to apply without thinking. For this reason, it is the basis for many approximate calculations in practice[1].

I will describe Riemann sums more formally and precisely in a moment, but first I will review a bit of notation that I will need.

4 Review of Σ notation

This may be familiar from algebra in which case take this as review (or skip it). Otherwise, this will be a fairly brief introduction.

The purpose of "Σ notation" is to have a fairly efficient way to describe sums of many terms. For example, consider the following sum.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11$$

Rather than write the whole sum beginning to end, we can observe that it has a common structure: we're just adding up consecutive numbers from 1 to 11. The standard way to write a sum like this is the following.

$$\sum_{k=1}^{11} k$$

[1]There are many improvements to this method that we won't discuss in 1A. They may appear in 1B depending on the tastes of the course head. These improvements include the "Trapezoid method," which is really nothing but averaging the results of the left-hand and right-hand sums, and a more sophisticated method called Simpson's rule. More generally, you can improve the method by detecting places where the curve is more wild (in this case, around 11am for example) and adding extra rectangles at those places to better capture the complicated behavior.

This notation means the following: we're summing some numbers, all of which have exactly the same form: some number k. This number k takes 11 possible consecutive values: these values start at $k = 1$ (as indicated in the subscript) and end at $k = 11$ (as indicated in the superscript). Here is a slightly different example: consider this notation.

$$\sum_{k=1}^{12} k^2$$

This means: I'm summing some numbers, one for integer value of k from 1 to 12 (this I learn from the subscript and superscript). Each number has the same form: it the square of some number k. Which number k? Well, it will be 12 different things: it will be 1, then 2, then 3, and so forth. I let k be each of these values, and add the results. So this Σ notation is identical to writing the sum out in full as follows.

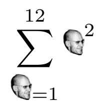
$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2$$

One point to emphasize here: the variable k is what is called a *dummy variable*. It has no particular meaning, except that it functions as a *counter* to keep track of where I am in the sum. The sum is meant to be a sum of some numbers squared, and k serves as a counter for which number is currently being squared.

I could use any symbol I like for this "dummy variable;" it's entire role is to stand in for what number it is that I am currently squaring. For example, you could write this if you like. All of the following expression means the same thing and are equally acceptable.

$$\sum_{n=1}^{12} n^2$$

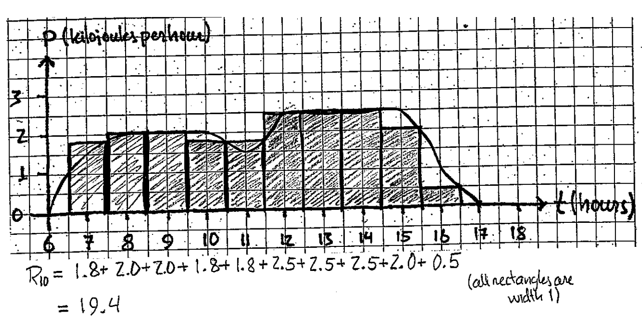
$$\sum_{\Xi=1}^{12} \Xi^2$$



5 Riemann sums, in full notation

Let's return to Riemann sums. For now I'll focus on right-hand sums. The goal of this section is to introduce sufficient notation to write a precise expression (that could be entered into a computer) that describes what a Riemann sum is.

Consider the sum R_{10} from before.



There are 10 rectangles here, with endpoints at various different values of x. Let's give these values of x the following names.

$$\begin{aligned}
x_0 &= 6.5 \\
x_1 &= 7.5 \\
x_2 &= 8.5 \\
x_3 &= 9.5 \\
x_4 &= 10.5 \\
x_5 &= 11.5 \\
x_6 &= 12.5 \\
x_7 &= 13.5 \\
x_8 &= 14.5 \\
x_9 &= 15.5 \\
x_{10} &= 16.5
\end{aligned}$$

These x coordinates tell where each rectangle begins and ends.

- The first rectangle begins at x_0 and ends at x_1.
- The second rectangle begins at x_1 and ends at x_2.
- The third rectangle begins at x_2 and ends at x_3.
- $\cdots$
- The kth rectangle begins at x_{k-1} and ends at x_k.

Now, the width of every rectangle is the same: the total interval has length $16.5 - 6.5 = 10$, and we are using ten rectangles, so each has width $10/10 = 1$. This common width is called Δx. Note that $x_1 - x_0 = \Delta x$, $x_2 - x_1 = \Delta x$, and so forth.

Now what is the heigh of each rectangle? We are using the right endpoint. Therefore:

- The first rectangle (which ends at $x_1 = 7.5$) has height $P(x_1) = 1.8$.
- The second rectangle has height $P(x_2) = 2$.
- The third rectangle has height $P(x_3) = 2$.
- $\cdots$
- The kth rectangle has height $P(x_k)$.

Therefore, the kth rectangle has width $\Delta x = 1$, and height $P(x_k)$. So it has area $P(x_k) \cdot \Delta x$. If we want to add all these areas together, you just put a Σ in front and tell me which values of k you plan to use. In this case, I want the first, second, third, fourth, $\cdots$, and tenth rectangles, so I want $k = 1$ to $k = 10$. Therefore the right-hand Riemann sum can be written like this.

$$R_{10} = \sum_{k=1}^{10} P(x_k)\Delta x$$

To give this to a computer, I also need to tell you what Δx and x_k are: in this case they are just $\Delta x = 1$ and x_k are the numbers listed above.

In general, suppose we want to approximate the following area:

The area under the curve $y = f(x)$ from $x = a$ to $x = b$.

We can define right-hand Riemann sums R_n for any number n we like. It works just like above.

First, decide on the width Δx of the rectangles. There should be N of them, and they should equally split up $[a, b]$, so each should have length $\Delta x = \frac{b-a}{n}$. Their endpoints should be given like this.

- $x_0 = a$
- $x_1 = a + \Delta x$
- $x_2 = a + 2\Delta x$
- $\cdots$
- $x_k = a + k\Delta x$
- $\cdots$
- $x_n = a + n\Delta x = b$

More briefly, we can just write $x_k = a + k\Delta x$; this includes both $x_0 = a$ and $x_n = b$ as special cases.

Then to take a right-hand sum, we will sum the areas of k rectangles. The kth rectangle starts at x_{k-1} and ends at x_k, so it has width $x_k - x_{k-1} = \Delta x$ and height $f(x_k)$ (the value of the function at the right endpoint). So the area of the kth rectange is $f(x_k)\Delta x$. And I just want to add all of these together. There are n for them: one for each value of k from 1 to n. So the right-hand sum can be written this way.

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x \quad \left(\text{where } \Delta x = \frac{b-a}{n} \text{ and } x_k = a + k\Delta x\right)$$

This is now a form you can ask a compute to compute for you: as long as the computer knows the function $f(x)$, and you define Δx and x_k in the way shown, it can simply compute all the areas of the rectangles and add them up.

I won't go through how to obtain the notation for left-hand sums in detail, but I will write the result. The only difference is that you should sum from $k = 0$ to $k = n-1$, since this has the effect of shifting which endpoint is put into $f(x)$ at each term in the sum.

$$L_n = \sum_{k=0}^{n-1} f(x_k)\Delta x \quad \left(\text{where } \Delta x = \frac{b-a}{n} \text{ and } x_k = a + k\Delta x\right)$$

6 Example: the area under $y = x^2$ from $x = 0$ to $x = 3$

Let's see how the notation above turns out when we consider the area under the parabola $y = x^2$ form $x = 0$ to $x = 3$. Let's say we have chosen some number n, and want the Riemann sum approximation with n terms. Then in the notion of the previous section, we have:

$$\begin{aligned}
f(x) &= x^2 \\
a &= 0 \\
b &= 3 \\
\Delta x &= \frac{3-0}{n} = \frac{3}{n} \\
x_k &= 0 + k \cdot \Delta x \\
&= k \cdot \frac{3}{n}
\end{aligned}$$

Therefore the right-hand sum for n rectangles is given by the following sum. All I have done is taken the equation from the previous section and substituted in the definitions of $f(x)$, x_k, and Δx for this particular case.

$$R_n = \sum_{k=1}^{n} \left(k \cdot \frac{3}{n}\right)^2 \cdot \frac{3}{n}$$

Let's simplify this expression a bit to get a sense for what this sum turns out to be.

$$\begin{aligned} R_n &= \sum_{k=1}^{n} \left(k \cdot \frac{3}{n}\right)^2 \cdot \frac{3}{n} \\ &= \sum_{k=1}^{n} k^2 \cdot \left(\frac{3}{n}\right)^3 \\ &= \left(\frac{3}{n}\right)^3 \sum_{k=1}^{n} k^2 \end{aligned}$$

At this point, we can actually evaluate the sum in closed form, using the following identity (we do not expect you to know this identity; to see why it is true, you can google "sum of first n squares" and find several accounts of this formula).

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Using this formula, we obtain the following closed form for R_n, which we can then re-express slightly as shown.

$$\begin{aligned} R_n &= \frac{3^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{3^3 \cdot 1 \cdot (1+1/n) \cdot (2+1/n)}{6} \\ &= \frac{3^3}{6}(1+1/n)(2+1/n) \end{aligned}$$

From this expression, we can see what happens when you take finer and finer approximations (with more and more rectangles): the terms $1/n$ go to 0, and the value of R_n will approach $\frac{3^3}{6} \cdot 2 = 9$.

In fact, the exact value of the area under x^2 from $x = 0$ to $x = 3$ is 9. One way to define this area is precisely as the limit of these Riemann sums. We will discuss this more in the next lecture.