Lecture 21: Logarithmic differentiation

Nathan Pflueger

28 October 2013

1 Introduction

Today we will discuss an important example of implicit differentiate, called logarithmic differentiation. The technique is very simple: to differentiate a function, it is often easier to first differentiate the logarithm of the function. Why this can be easier will be clear in examples: generally speaking this technique is most effective when the function is a complicated product or quotient of several functions.

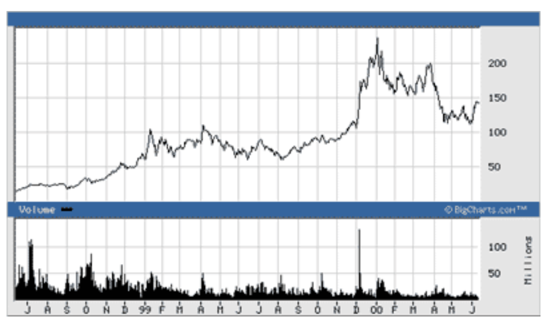
The derivative of the logarithm is also an important notion in its own right, used in many modeling situations. I'll begin be discussing one of these.

The reference for today is Stewart §3.7.

2 How should you measure changes in prices?

Note: if you don't like finance (and I don't blame you), you can replace "price" with "population" and "stock" with "country" in this section, and all the same remarks apply.

If you have studied any finance, you are probably aware that there are two rather different ways that a stock price (or exchange rate, commodity price, GDP, or other indicators) are graphed: either linearly or logarithmically. The following two pictures (borrowed from the Motley Fool[1]) show the difference.



[1] Detailed discussion at http://www.fool.com/foolfaq/foolfaqcharts.htm

The first plot is a linear plot: the price is plotted on the y axis as usual. In the second plot, the y coordinate is not the price itself, but rather some *logarithm* of the price. Notice that on the logarithmic plot, the vertical space between price 50 and 100 is the same as the distance between 100 and 200. This is because the difference in logarithms $\ln 100 - \ln 50$ and $\ln 200 - \ln 100$ are equal: they are both equal to $\ln 2$.

Why are logarithmic plots so important in finance? The reason is that what matters in investing is never the difference between two prices, but the ratio between them. If you buy at 50 and sell at 100, you double your money. It is no difference form buying at 100 and selling at 200. So the logarithmic chart more faithfully represents meaningful changes in price. Indeed, the top plot makes it look like the price is varying in a more volatile way at the very end, but the second plot dispels this illusion: it just looks like it moves more because the price itself is larger, so changes that are small as ratios are magnified. This illusion is removed on the second plot.

Now here is a question.

Question. Suppose that the price of a stock is $f(t)$, where t is the number of year since 1970. What is the most useful way to express how quickly the stock price is rising at a given moment?

Here are three possible answers, i.e. three possible ways to summarize how quickly the price is changing.

1. **The derivative** $f'(t)$. This certainly measure how fast the price is increasing, but it is actually *not a very useful measure.* Here's why: suppose that someone tells you "the stock price is increasing by 100 dollars a day!" What can you infer from this? Almost nothing. Because if the stock is Berkshire Hathaway (current price: $175,435$ dollars per share), this increase would be very small compared to the price of buying shares. On the other hand, if these are shares of Yahoo (current price: 32 dollars per share), an increase of 100 dollars would amount to being able to quadruple your money. So the derivative by itself doesn't convey useful information.

2. **The derivative, divided by the share price** $f'(t)/f(t)$. This tells you exactly how fast the price is increasing, *as a portion of the current price.* This is more relevant for an investor. For example, if $f'(t)/f(t) = 0.01$, that means that the price is incasing by about 1 percent per day.

3. **The derivative of the logarithm** $\frac{d}{dt} \ln f(t)$. As we saw above, you can see the meaningful motion of price better on a logarithmic plot, so we can just look at rates of change (slopes) on that plot instead.

The wonderful fact is that **the second two choices are completely identical.** Looking at rates of change on a logarithmic graph is *exactly the same* as taking the derivative and comparing it to the value of the function. You may have already realized why this is; we'll discuss this fact (with a proof) in the next section.

3 Logarithmic differentiation

The mathematical fact underlying my remarks at the end of last section is the following.

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

This fact is a consequence of the chain rule. Here's a quick derivation, using implicit differentiation.

$$\begin{aligned}
e^{\ln f(x)} &= f(x) \quad \text{definition of } \ln \\
\frac{d}{dx} e^{\ln f(x)} &= f'(x) \quad \text{differentiate both sides} \\
e^{\ln f(x)} \frac{d}{dx} \ln f(x) &= f'(x) \quad \text{chain rule on the left} \\
f(x) \frac{d}{dx} \ln f(x) &= f'(x) \quad \text{substitute } e^{\ln f(x)} = f(x) \\
\frac{d}{dx} \ln f(x) &= \frac{f'(x)}{f(x)}
\end{aligned}$$

Alternatively, if you just know the fact that $\frac{d}{dx} \ln x = \frac{1}{x}$, then this equation follows from the chain rule. In my section, we differentiated $\ln x$ a couple weeks ago; for other sections, you can view this as a special case of the equation above, in the case $f(x) = x$.

Sometimes, you will hear people call the function $\frac{f'(x)}{f(x)}$ called the "logarithmic derivative" or the "log-derivative" of $f(x)$. It is an important notion in finance (for reasons mentioned in the last section), as well as many parts of pure and applied mathematics. Its main use for us will be that it turns out to be easier to compute in many cases than the ordinary derivative (and we can obtain the ordinary derivative directly from it!). Here is an example, which you explored experimentally in your homework.

Example 3.1. We will differentiate the function $f(x) = x^x$ (for $x > 0$). To do this, we will first take the logarithm of both sides, and then differentiate implicitly, using the fact at the beginning of this section. Here's how it works.

$$\begin{aligned}
f(x) &= x^x \quad \text{(definition)} \\
\ln f(x) &= x \ln x \quad \text{(log of both sides)} \\
\frac{f'(x)}{f(x)} &= \frac{d}{dx}(x \ln x) \quad \text{(derivative of both sides)} \\
&= \ln x + x \frac{d}{dx} \ln x \\
&= \ln x + 1 \\
f'(x) &= (\ln x + 1) f(x) \quad \text{(multiply by } f(x)) \\
f'(x) &= (\ln x + 1) x^x \quad \text{(re-substitute the definition of } f(x))
\end{aligned}$$

Therefore we conclude that $\frac{d}{dx} x^x = (\ln x + 1) x^x$.

This basic strategy is often useful to evaluate derivatives. Sometimes these are derivates that we would otherwise have no way to evaluate; at other times this gives a much easier way to evaluate something that we could do in principle, but is otherwise rather difficult. Many examples are in the next section.

Reminder: although we won't discuss it much in this class, remember (for the future) that in many cases the log-derivative $f'(x)/f(x)$ is more useful that the derivative, in and of itself (especially in finance), so often you should just compute it and stop there.

4 Examples

One of the main uses for logarithmic differentiation is to differentiate functions of the form $f(x)^g(x)$, as in the following three examples.

Example 4.1. DIfferentiate $f(x) = (\cos x)^{\sin x}$.

Solution. Take the logarithm of both sides and differentiate; then solve for $f'(x)$.

$$\begin{aligned}
f(x) &= (\cos x)^{\sin x} \\
\ln f(x) &= \sin x \cdot \ln(\cos x) \\
\frac{f'(x)}{f(x)} &= \cos x \ln(\cos x) + \sin x \cdot \frac{-\sin x}{\cos x} \\
&= \cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x} \\
f'(x) &= f(x)\left(\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right) \\
f'(x) &= (\cos x)^{\sin x + 1} \ln(\cos x) - (\cos x)^{\sin x - 1} \sin^2 x
\end{aligned}$$

Example 4.2. Differentiate $x^{\sqrt{x}} + 7^x$.

Solution. It would not be helpful to try to differentiate this function *as a whole* with logarithmic differentiation. Instead, differentiate the two summands separately. First, deal with $x^{\sqrt{x}}$, as follows.

$$\begin{aligned}
\ln(x^{\sqrt{x}}) &= \sqrt{x} \ln x \\
\frac{(x^{\sqrt{x}})'}{x^{\sqrt{x}}} &= \frac{d}{dx}\left(\sqrt{x} \ln x\right) \quad \text{(differentiate both sides)} \\
&= \frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \cdot \frac{1}{x} \\
&= \frac{1}{2\sqrt{x}} \ln x + \frac{1}{\sqrt{x}} \\
&= \frac{\ln x + 2}{2\sqrt{x}} \\
(x^{\sqrt{x}})' &= x^{\sqrt{x}} \cdot \frac{\ln x + 2}{2\sqrt{x}} \\
&= \frac{1}{2} x^{\sqrt{x} - \frac{1}{2}} (\ln x + 2)
\end{aligned}$$

Note. This function is complicated enough that there isn't any one good way to simplify it. Do not worry too much about simplifying things like this. I will sometimes simplify them in the notes for the sake of clarity, but we will not generally expect you to be able to do this.

We can also differentiate 7^x using logarithmic differentiation (or you can simply recall that for any constant b, $\frac{d}{dx} 7^x = \ln 7 \cdot 7^x$, we we showed earlier using the chain rule):

$$\begin{aligned}
\ln(7^x) &= x \ln 7 \\
\frac{(7^x)'}{7^x} &= \ln 7 \\
(7^x)' &= \ln 7 \cdot 7^x
\end{aligned}$$

Therefore we can differentiate the sum as follows, by combining these two.

$$\frac{d}{dx}(x^{\sqrt{x}} + 7^x) = \frac{1}{2} x^{\sqrt{x} - \frac{1}{2}} (\ln x + 2) + \ln 7 \cdot 7^x$$

Example 4.3. Differentiate $(\ln x + x)^{\sqrt{x}}$ (do not try to fully simplify).

Solution.

$$\begin{aligned}
f(x) &= (\ln x + x)^{\sqrt{x}} \\
\ln f(x) &= \sqrt{x} \ln(\ln x + x) \\
\frac{f'(x)}{f(x)} &= \frac{1}{2\sqrt{x}} \ln(\ln x + x) + \sqrt{x} \frac{d}{dx} \ln(\ln x + x) \quad \text{(product rule)} \\
&= \frac{1}{2\sqrt{x}} \ln(\ln x + x) + \sqrt{x} \frac{\frac{d}{dx}(\ln x + x)}{\ln x + x} \quad \text{(chain rule)} \\
&= \frac{1}{2\sqrt{x}} \ln(\ln x + x) + \sqrt{x} \frac{\frac{1}{x} + 1}{\ln x + x} \\
f'(x) &= f(x) \left(\frac{1}{2\sqrt{x}} \ln(\ln x + x) + \sqrt{x} \frac{\frac{1}{x} + 1}{\ln x + x} \right) \\
f'(x) &= (\ln x + x)^{\sqrt{x}} \cdot \left(\frac{1}{2\sqrt{x}} \ln(\ln x + x) + \sqrt{x} \frac{\frac{1}{x} + 1}{\ln x + x} \right)
\end{aligned}$$

It would not be very illuminating to simplify this. It is just an ugly function.

The examples above would be impossible to differentiate with the methods we've studied so far. There are also many situations where you can use other techniques, but logarithmic differentiation is significantly easier to carry out. As usual, there are no hard, fast rules about this, but as a general principle: *if the function is built up from multiplying and dividing several simpler functions, it will be easier to differentiate logarithmically.* The reason is that the logarithm can be used to split a product or quotient into a sum or a difference.

Example 4.4. As a first example, here is a **fast derivation of the product and quotient rules.**

Product rule: suppose that f and g are two functions of x (I'll abbreviate $f(x)$ with f for readability):

$$\begin{aligned}
\ln(fg) &= \ln f + \ln g \quad \text{(properties of logarithms)} \\
\frac{(fg)'}{fg} &= \frac{f'}{f} + \frac{g'}{g} \quad \text{(differentiate)} \\
(fg)' &= fg(\frac{f'}{f} + \frac{g'}{g}) \\
&= f'g + g'f
\end{aligned}$$

Quotient rule:

$$\begin{aligned}
\ln\left(\frac{f}{g}\right) &= \ln f - \ln g \\
\frac{(f/g)'}{(f/g)} &= \frac{f'}{f} - \frac{g'}{g} \\
\left(\frac{f}{g}\right)' &= \frac{f}{g}\left(\frac{f'}{f} - \frac{g'}{g}\right)
\end{aligned}$$

I leave it to you to see that a bit of elementary algebra gives the usual statement of the quotient rule from here. For me, this is a much easier way to remember why the quotient rule is true than any other mnemonic device.

Example 4.5. Differentiate $f(x) = \dfrac{(x+2)^{27}}{e^x\sqrt{x^2+167}}$ (you do not need to fully simplify).

Solution. This is a function that you certainly can differentiate by means we've studied before. However, because it is built by by several multiplications and divisions, it will be vastly simplified by logarithmic differentiation. The computation is shown below.

$$
\begin{aligned}
f(x) &= \frac{(x+2)^{27}}{e^x\sqrt{x^2+167}} \\
\ln f(x) &= 27\ln(x+2) - x - \frac{1}{2}\ln(x^2+167) \\
\frac{f'(x)}{f(x)} &= \frac{27}{x+2} - 1 - \frac{1}{\cancel{2}}\frac{\cancel{2}x}{x^2+167} \\
f'(x) &= \frac{(x+2)^{27}}{e^x\sqrt{x^2+167}}\left(\frac{27}{x+2} - 1 - \frac{x}{x^2+167}\right)
\end{aligned}
$$

I do not consider it advisable to "simplify" from here (it will probably only get messier).