Lecture 2: Rates of change

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1 Introduction

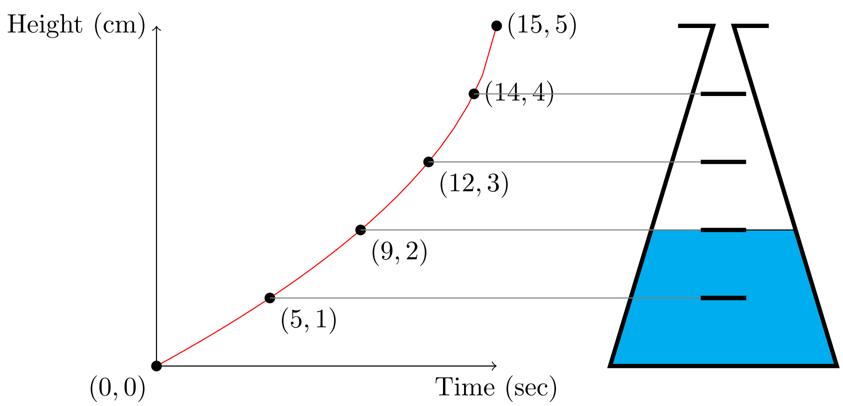
Calculus was initially developed in large part to study the motion of planets and other physical problems. The basic insight behind Newton's laws of motion is that the motion of physical objects is best describing by talking about their velocity and acceleration; calculus provided the vocabulary and logical apparatus to do this. Velocity is the most familiar example of a *rate of change*, while accelerations is the rate of change of velocity.

This lecture begins a basic, informal discussion about what we mean when we say *rate of change*. The basic idea is simple: it tells how much a quantity (like distance traveled along a road) will change in a given period of time. We will see how this concept graphically as a secant line. The trouble is that objects don't keep moving at a steady speed forever, so we must think about what a rate of change (like velocity) can mean *at a particular instant*, not over a period of time. Graphically this corresponds to the idea of a *tangent line,* which is a sort of idealized secant line. This will lead into a discussion of limits in the next lecture, where we will describe the modern way to make this transition (secant to tangent) a bit more precise.

The reference for todays material is section 2.1 of Stewart.

2 Example: rising water level in a flask

Consider the "conical flask" mentioned in the bottle-filling worksheet (I will explain the situation here; it is not necessary for you to look up this worksheet). The flask shown at the right is filling up with water at a constant rate. The graph on the left shows how high the water level is at various times (the flask has four evenly spaced marks, corresponding to four of the labeled points on the graph). Because the flask gets narrower at the top, it takes less and less water for each centimeter that the water level rises; thus the graph gets steeper and steeper.



Question. How quickly, on average, does the water level rise as the bottle fills (in centimeters per second)?

Answer. The water level rises a total of 5cm, over the course of 15 seconds. Therefore the average rate that the water rises is $\frac{5cm}{15s} \approx 0.333 cm/s$.

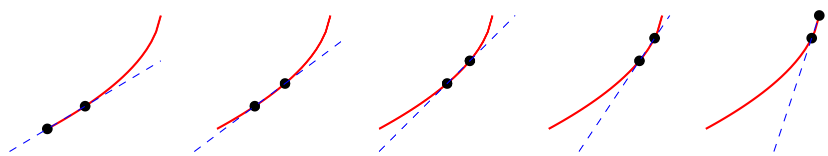
Question. How quickly (on average) is the height rising as it rises from the bottom of the bottle to the first mark (1 cm up)? From the first mark to the second? The second to the third? The third to the fourth? From the fourth to the top of the bottle?

Answer. In each case, the average rate that the water rises is the total rise (in centimeters) divided by the time (in seconds). Therefore these are:

- From the bottom to the first mark: $\frac{1cm}{5s} = 0.200 cm/s$.
- From the first mark to the second: $\frac{(2-1)cm}{(9-5)s} = 0.250 cm/s$.
- From the second mark to the third: $\frac{(3-2)cm}{(12-9)s} \approx 0.333 cm/s$.
- From the third mark to the fourth: $\frac{(4-3)cm}{(14-12)s} = 0.500 cm/s$.
- From the fourth mark to the top: $\frac{(5-4)cm}{(15-14)s} = 1.000 cm/s$.

As we could guess from looking at the picture, the rate that the height goes up is increasing as the bottle fills. It increases gently at first and then begins to increase dramatically at the end.

Visually, these rates of increase correspond to the slopes of the secant lines joining adjacent pairs of points. Note that in this pictures, the x and y axes use slightly different scales; this is why the line look steeper than you would think from the numbers.



$0.200 cm/s$ $\quad$ $0.250 cm/s$ $\quad$ $0.333 cm/s$ $\quad$ $0.500 cm/s$ $\quad$ $1.000 cm/s$

3 Example: a bicycle speedometer

Many bicycle speedometers work in the following way: a small magnet is attached to one of the spokes of a wheel. Every time the wheel makes a full revolution, a small sensor detects the magnet passing by. How can this device be used to measure the speed the bicycle is traveling?

This device can measure speed because it knows the two main pieces of information: the number of meters traveled, and the amount of time it took to travel them. The distance traveled is just the number of times the magnet passed by multiplied by the circumference of the wheel (since this is how far the bike travels with each revolution). The device can determine the time it takes to travel this distance by measuring the time between each moment it detects the magnet.

Suppose that the bicycle wheel has circumference 2 meters. Then the speedometer, in effect, can tell how much time has passed every time the bicycle travels another two meters. Suppose that it records the following data.

Time elapsed (seconds)	0.00	2.0	3.6	4.8	5.6	6.0
Distance traveled (meters)	0	2	4	6	8	10

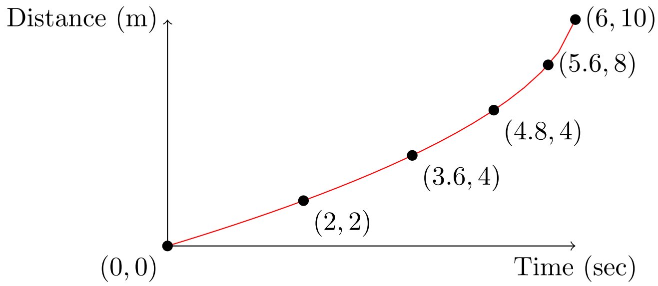
Question. At what speed does the bicycle travel these 10 meters, on average? How fast is it traveling for the first 2 meters? The second 2 meters? The third, fourth, and fifth?

Answer. The bicycle travels the first 2 meters at an average of $\frac{2m}{2s} = 1m/s$ (this is approximately $2.24 miles per hour$). The second two meters are completed at $\frac{2m}{1.6s} = 1.25 m/s$. The third are completed at

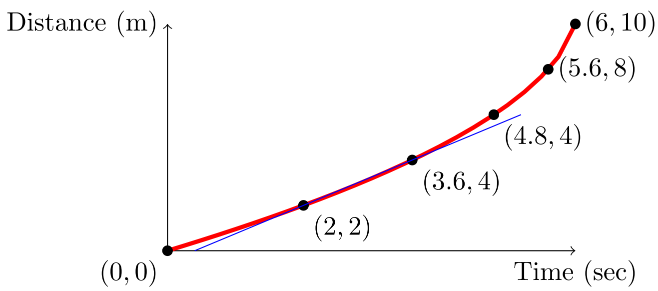
$\frac{2m}{1.2s} \approx 1.67m/s$. The fourth two meters are completed at $\frac{2m}{0.8s} = 2.50m/s$. The last two meters are completed at $\frac{2m}{0.4s} = 5m/s$, or about 11 miles per hour.

Notice that **this example is essentially identical to the previous example.** All I've done is multiplied the x values by 0.4 and the y values by 2 to get more realistic figures for this situation.

Question. How fast is the bicycle traveling after exactly 2 seconds?

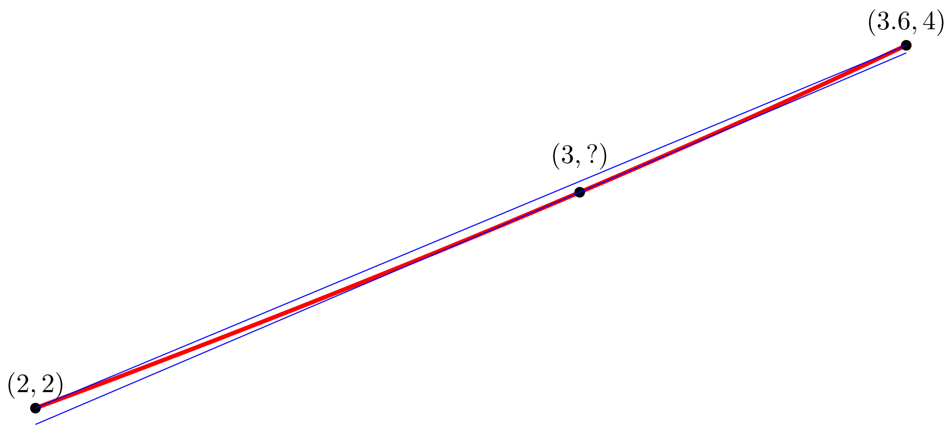


Answer. The tricky thing about this question is that we're no longer asking for an average – we want to know how fast the bicycle is traveling *at that instant*. And unfortunately the answer is: the device can't tell precisely. All it knows is how often the magnet goes all the way around. However, we can get a pretty good guess: we know that the instant when 3 seconds has passed lies somewhere in between the moment when the wheel made one revolution (2 seconds in) and the moment when the wheel made two revolutions (3.6 seconds in). From the previous problem, we know that the average speed in this interval was **1.25 m/s**. This is the slope of the tangent line between the two sensor readings mentioned.



From the picture, we see that the secant line between the corresponding points on the graph is a very good approximation of the function in this range, so its slope is a very good guess at the instantaneous rate of change. To get a better guess would require either a more precise measuring device (for example, magnets on more spokes of the wheel) or some assumptions about how quickly the bicycle accelerates during this interval.

To even be able to tell that the secant line is not a perfect approximation requires zooming in on this graph, as shown below. Here, both the secant line we've used to approximate the speed, and the tangent line whose slope is the true speed, are shown. The difference in their slopes is clearly very small.



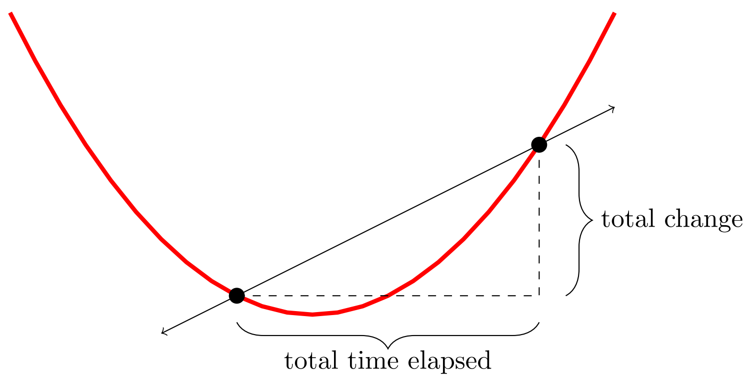
Question to think about. Is this sort of bike speedometer more accurate at high speeds or low speeds? Why?

4 Secant lines and tangent lines

As we've seen in the last two examples, the concept of *average rate of change*, namely

$$\text{Avg. rate of change} = \frac{\text{total change}}{\text{total time elapsed}}$$

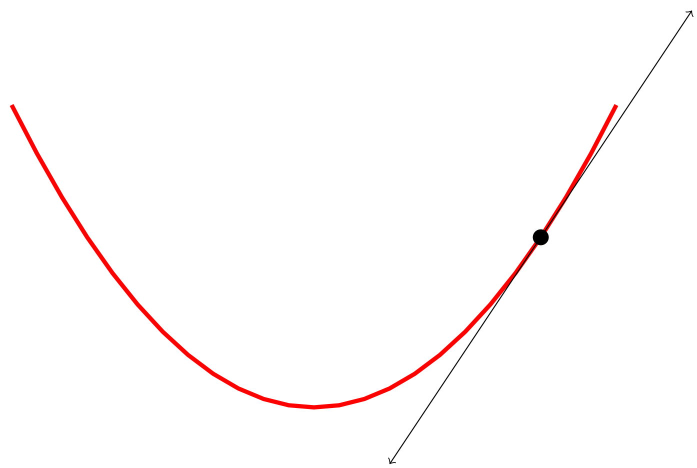
has a geometric counterpart: the slope of a secant line. This is illustrated in the following picture. A secant line just means a line passing through two chosen points on a graph.



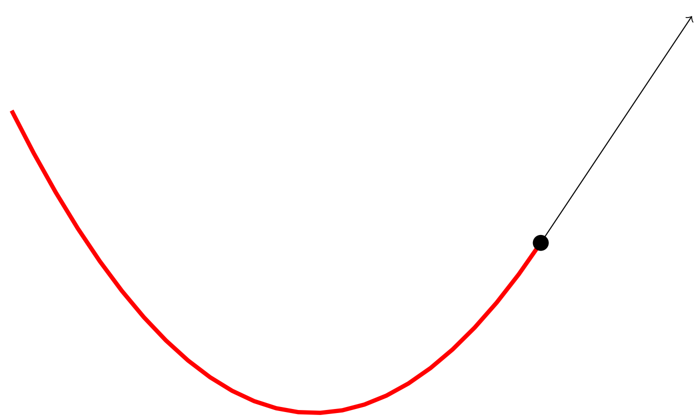
A slightly more slippery concept is the *instantaneous rate of change.* It is remarkably hard to define what we mean by this term. But we have all watched the speedometer on a car: at any moment, it is pointing *somewhere.* Where it's pointing, as far as we are concerned, is the speed that the car is currently, *at that very instant,* traveling.

The graphical equivalent of instantaneous rate of change is the *tangent line.* One way to understand the tangent line is purely visually: it is the line that perfectly kisses the graph at a chosen point[1].

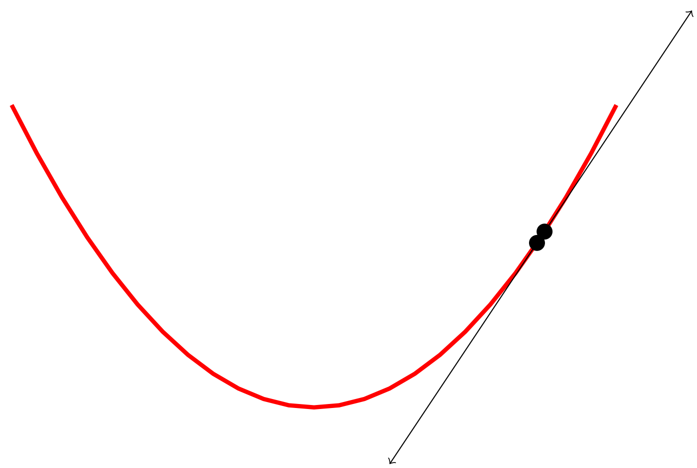
[1]A use the word kisses here partly for historical reasons: sometimes the tangent line is called an *osculating line*, from the latin word *osculum*, for kiss.



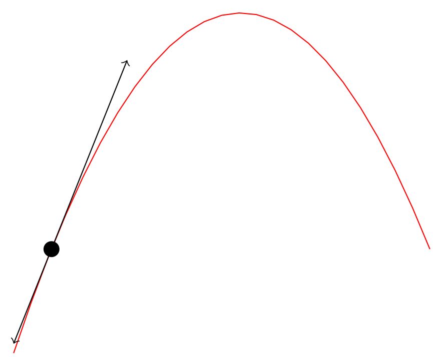
Why should such a line, however nicely it kisses the graph, tell something like an instantaneous rate of change? One explanation is that it shows where the graph would go if it would just settle down, stop turning, and travel straight. The slope, then, is the average rate of change after the points where the graph has started moving straight.



Another way to understand the tangent line is that it is a sort of *idealized secant line*. We like secant lines better if the two points are very close together, so that we can measure speed clearly without too much acceleration happening. There is no perfect secant line, because the two points are always a little apart. But we can try harder and harder, and draw closer and closer. The tangent line is a sort of Platonic ideal: the secant line that God would draw, where the two points are so close together that they are actually the same point.



Example 4.1. What is the slope of the tangent line to the graph of $y = -x^2 + 12x - 11$ at $(1, 0)$?



Solution. Begin by considering any other point (x, y) on the graph, and find the slope of the secant line through $(1, 0)$ and (x, y). The slope of this line is $\frac{\text{rise}}{\text{run}} = \frac{y-0}{x-1}$. The following table shows some sample values, with x getting closer and closer to 1.

x	y	slope
2	9	$\frac{9-0}{2-1} = 9$
1.1	0.99	$\frac{0.99}{0.1} = 9.9$
1.01	0.0999	$\frac{0.0999}{0.01} = 9.99$
1.001	0.009999	$\frac{0.009999}{0.001} = 9.999$
1	0	$\frac{0}{0} = \mathbf{???}$
0.999	-0.010001	$\frac{-0.010001}{-0.001} = 10.001$
0.99	-0.1001	$\frac{-0.1001}{-0.01} = 10.01$
0.9	-1.01	$\frac{-1.01}{-0.1} = 10.1$
0.5	-5.25	$\frac{-5.25}{-0.5} = 10.5$

The pattern is clear: for values of x very close to 1, the secant line from $(1, 0)$ to (x, y) has slope very close to 10. But we can't quite move the second point "right on top of" the first point, since the slope would then appear to be $0/0$, which is undefined.

In this case, however, we can see fairly easily why the slope gets close to 10 in such a regular way. The value of the slope, for any value of x other than 1, is given by

$$
\begin{aligned}
\frac{\text{rise}}{\text{run}} &= \frac{y}{x-1} \\
&= \frac{-x^2+12x-11}{x-1} \\
&= \frac{(x-1)(11-x)}{x-1} \\
&= 11-x.
\end{aligned}
$$

Now *this* is an expression that we can put 1 into very easily, to obtain 10. This suggests that if we *could* take a secant line "from the point to itself," the slope would be 10. And indeed, this is the slope of the tangent line.

The next lecture discusses the usual formalism for doing problems like the one above.

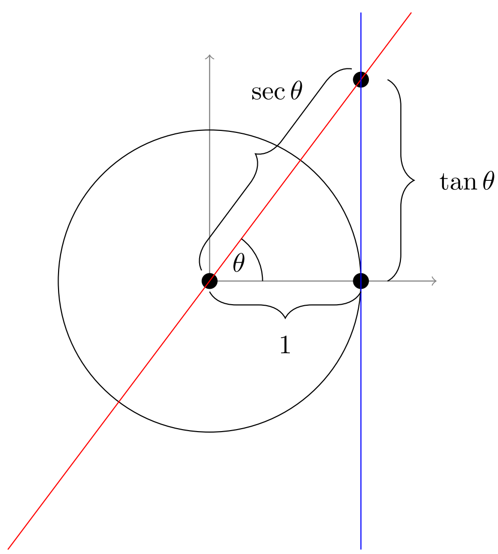
5 Appendix: The other use of the words "secant" and "tangent"

Note. Whenever a section is labeled as an appendix in these notes, it is *completely optional* and included just for those who may be interested. Of course, all of these notes are optional reading, but the appendices in particular are not about anything that will be mentioned in homework or exams.

In class, a student asked about what the terms secant and tangent (as we use them here) have to do with their usual usage, namely the following trigonometric functions.

secant function	$\sec\theta = \frac{1}{\cos\theta}$
tangent function	$\tan\theta = \frac{\sin\theta}{\cos\theta}$

The answer is that the two things (secant/tangent lines, and the secant/tangent functions) are rooted in the same etymology, but don't have any substantive relationship that you need to worry about. The common etymology comes from the following diagram.



In the early days of trigonometry, functions were defined very explicitly in terms of lengths of line segments in figures. The functions $\sec\theta$ and $\tan\theta$ were defined like in the figure above: two lines are drawn to a circle of radius 1: one of them is a tangent line (blue), and one is a secant line (red). The angle θ is as shown. Then the quantities $\sec\theta$ and $\tan\theta$ are defined as the two lengths shown.

So the common name simply comes from the fact that the secant function is defined in terms of a segment of a secant *line* to a circle, while the tangent function is defined to be a segment of a tangent line.