Review problems

Note: these answers are very brief, but model how much explanation is acceptable.

Part I: fast recall (no explanations needed)

Diff. rules

(1) Differentiate:

a) x^{27} $\quad 27x^{26}$

b) $\sqrt{x}$ $\quad \frac{1}{2\sqrt{x}}$

c) $1/x$ $\quad -1/x^2$

d) e^x $\quad e^x$

e) $\ln x$ $\quad 1/x$

f) $\cos x$ $\quad -\sin x$

g) $\tan x$ $\quad \sec^2 x$

h) $\tan^{-1} x$ $\quad 1/(1+x^2)$

i) $\ln(\sin x)$ $\quad \cos x/\sin x$ or $\cot x$

j) $x \cdot e^x$ $\quad e^x + x \cdot e^x$

Linear approx.

(2) Estimate $\sin(0.02)$ w/ linear ~~approximation~~.

$\sin x \approx x$ near $x=0$

$\Rightarrow \boxed{\sin(0.02) \approx 0.02}$

optimization

(3) Find the ~~maximum and~~ minimum of $f(x) = e^x - x$ on $(-\infty, \infty)$.

$f'(x) = e^x - 1$

$f'(x) = 0 \Leftrightarrow x = 0$

$f''(x) = e^x$

$\Rightarrow$ concave up.

so $x=0$ is the absolute minimum: $\boxed{(0,1)}$

(4) Find the maximum and minimum of $f(x) = x^3 - 3x$ on $[0,2]$.

$f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x+1)(x-1)$

$f''(x) = 6x$

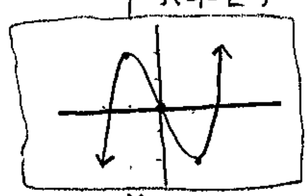
crit. numbers $-1, 1$ (only 1 in $[0,2]$)

$f(0)=0$, $f(1)=-2$, $f(2)=2$ — crits. & end-points

$\boxed{\text{max: } (2,2) \quad \text{min: } (1,-2)}$

graphing

(5) Sketch the graph of $f(x) = x^3 - 3x$.



(6) Evaluate $\lim_{x\to-\infty} \frac{e^x+1}{e^x-1}$ and $\lim_{x\to\infty} \frac{e^x+1}{e^x-1}$.

$= \frac{0+1}{0-1} = \boxed{-1}$

$\lim_{x\to\infty} \frac{e^x+1}{e^x-1} = \lim_{x\to\infty} \frac{1+e^{-x}}{1-e^{-x}} = \frac{1+0}{1-0} = \boxed{1}$

related rates

(7) Suppose f, g are functions such that $f(t)g(t) = t$ for all t, $f(0) = 7$ $g(0) = 0$ and $f'(0) = 1$. Find $g'(0)$.

$f'g + g'f = 1$

$g'f = 1 - f'g$

$g' = \frac{1-f'g}{f}$

$g'(0) = \frac{1-1\cdot 0}{7}$

$= \boxed{1/7}$

Part II : longer problems.

(8) (Differentiation) Find derivatives:

a) $\ln(\ln(\ln(x)))$

$$\frac{d}{dx}\left(\ln(\ln(\ln(x)))\right) = \frac{1}{\ln(\ln(x))}\cdot\frac{d}{dx}\left(\ln(\ln(x))\right)$$

$$= \frac{1}{\ln(\ln(x))}\cdot\frac{1}{\ln(x)}\cdot\frac{d}{dx}\ln x = \boxed{\frac{1}{\ln(\ln(x))\cdot\ln(x)\cdot x}}$$

b) $\dfrac{e^x\sqrt{\sin x}}{(x+1)^7(x-2)^9}$

$\ln f(x) = x + \frac{1}{2}\ln(\sin x) - 7\ln(x+1) - 9\ln(x-2)$

$f'(x)/f(x) = 1 + \frac{1}{2}\cdot\frac{\cos x}{\sin x} - \frac{7}{x+1} - \frac{9}{x-2}$

$$\boxed{f'(x) = \frac{e^x\sqrt{\sin x}}{(x+1)^7(x-2)^9}\cdot\left[1+\frac{1}{2}\frac{\cos x}{\sin x} - \frac{7}{x+1} - \frac{9}{x-2}\right]}$$

c) $\sec^{-1}x$ (for $x>0$)

$f(x)=\sec^{-1}x$

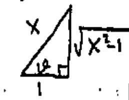
$\sec f(x) = x$

$\sec f(x)\cdot\tan f(x)\cdot f'(x) = 1$

$f'(x) = \dfrac{1}{\sec(\sec^{-1}x)\cdot\tan(\sec^{-1}x)}$

$\boxed{f'(x) = \dfrac{1}{x\cdot\sqrt{x^2-1}}}$ for $x>0$

using



$\tan(\sec^{-1}x) = \sqrt{x^2-1}$ for $x>0$

Note: for $x<0$, $\tan(\sec^{-1}x) = -\sqrt{x^2-1}$ so $f'(x) = -\dfrac{1}{x\sqrt{x^2-1}}$; in general $f'(x) = \dfrac{1}{|x|\cdot\sqrt{x^2-1}}$ (not needed here)

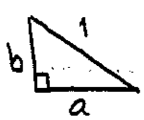
d) $e^{x^2}\cdot\ln(7x+1)$

Product rule:

$f'(x) = \boxed{(2xe^{x^2})\ln(7x+1) + e^{x^2}\cdot\frac{7}{7x+1}}$

(9) (optimization)

a) Find the maximum area of a right triangle w/ hypotenuse of length 1.



$a^2+b^2=1$

$b=\sqrt{1-a^2}$

note: $0\le a\le 1$

area $= \frac{1}{2}ab$

$f(a) = \frac{1}{2}a\sqrt{1-a^2}$ on $[0,1]$

$f'(a) = \frac{1}{2}\cdot 1\cdot\sqrt{1-a^2} + \frac{1}{2}a\cdot\frac{-2a}{2\sqrt{1-a^2}}$

$f'(a) = \frac{1}{2}\sqrt{1-a^2} - \frac{1}{2}\frac{a^2}{\sqrt{1-a^2}}$

$f'(a) = 0 \iff \frac{1}{2}\sqrt{1-a^2} = \frac{1}{2}\frac{a^2}{\sqrt{1-a^2}}$

$\iff 1-a^2 = a^2$

$\iff a = \pm\sqrt{1/2}$

$f(0)=0$, $f(\sqrt{\tfrac{1}{2}}) = \frac{1}{4}$, $f(1)=0$

$\Rightarrow \boxed{\max = 1/4}$

b) Find the maximum perimeter of ~~such a~~ a right triangle with hypotenuse of length 1.

$a^2+b^2=1$

$b=\sqrt{1-a^2}$

$0\le a\le 1$

perimeter $= a+b+1$

$f(a) = a+\sqrt{1-a^2}+1$

$f'(a) = 1 - \frac{a}{\sqrt{1-a^2}}$

$f'(a)=0 \iff \frac{a}{\sqrt{1-a^2}} = 1$

$\iff a^2 = 1-a^2$

$\iff a = \pm\sqrt{1/2}$

$f(0)=2$

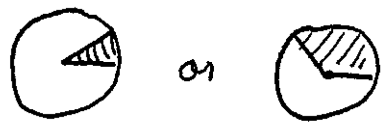
$f(\sqrt{\tfrac{1}{2}}) = \sqrt{2}+1$

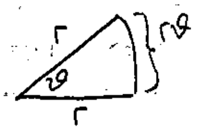
$f(1)=2$

so $\boxed{\max = \sqrt{2}+1}$

(10) (optimization)

Find the minimum perimeter of a slice of pie of area $50\ cm^2$.





$$\text{Area} = \tfrac{1}{2}\vartheta r^2 = 50$$
$$\text{Perimeter} = 2r + r\vartheta$$

$$\Rightarrow \vartheta = 50/\tfrac{1}{2}r^2 = 100/r^2$$

$$P(r) = 2r + r\cdot\frac{100}{r^2}$$
$$= 2r + \frac{100}{r}$$
$$P'(r) = 2 - 100/r^2$$
$$P''(r) = 200/r^3$$
$\Rightarrow$ con.up for $r>0$.

want to minimize $P(r)$ on $r>0$.
concave up $\Rightarrow$ local min will be absolute.

$$P'(r) = 0 \iff \frac{100}{r^2} = 2$$
$$\iff r = \sqrt{50}$$

so minimum is $P(\sqrt{50})$
$$= 2\sqrt{50} + \frac{100}{\sqrt{50}}$$ //could leave //answer in //this form.
$$= 10\sqrt{2} + 10\sqrt{2}$$
$$\boxed{= 20\sqrt{2}}$$

(11) (linear approximation)

a) Approximate $\tan^{-1}(1.03)$ (answer in terms of π).

$$f(x) = \tan^{-1}x$$
$$f'(x) = \frac{1}{1+x^2}$$

Approx. near the known value $x=1$:

$$\left.\begin{array}{l} f(1) = \pi/4 \\ f'(1) = 1/2 \end{array}\right\} \quad f(x) \approx \frac{\pi}{4} + \frac{1}{2}(x-1)$$
$$\Rightarrow \boxed{\tan^{-1}(1.03) \approx \frac{\pi}{4} + \frac{1}{2}\cdot 0.03} \quad \text{or } \frac{\pi}{4} + 0.015.$$

b) Find the tangent line to $y^2 = x^3 - x$ at the point $(2, \sqrt{6})$.

$$2y\cdot\frac{dy}{dx} = 3x^2 - 1$$
$$\frac{dy}{dx} = \frac{3x^2-1}{2y}$$

At $(2, \sqrt{6})$.
$$\frac{dy}{dx} = \frac{3\cdot 4 - 1}{2\sqrt{6}} = \frac{11}{2\sqrt{6}}$$

t. line:
$$\boxed{(y - \sqrt{6}) = \frac{11}{2\sqrt{6}}\cdot(x-2)}$$

(12) (graphing) Sketch the graph of $f(x) = x \cdot e^{-x^2}$
you may assume that $\lim_{x \to \pm\infty} f(x) = 0$ (we haven't covered how to evaluate these limits in detail).

- Determine local extrema, and where its increasing/decreasing.
- Determine where its concave up/down.

$$f'(x) = e^{-x^2} + x(-2x)e^{-x^2}$$
$$= (1-2x^2)e^{-x^2}$$

$$f''(x) = (-4x)e^{-x^2} + (1-2x^2)(-2x)e^{-x^2}$$
$$= (-6x+4x^3)e^{-x^2}$$
$$= -2x(3-2x^2)e^{-x^2}$$

extrema: $f'(x)=0 \iff 1=2x^2 \iff x=\pm 1/\sqrt{2}$

increasing $\iff 1>2x^2 \iff x<-1/\sqrt{2}$ or $x>1/\sqrt{2}$

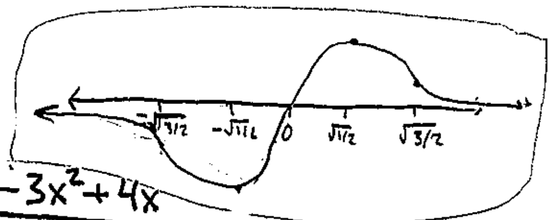
decreasing $\iff 1<2x^2 \iff -1/\sqrt{2}<x<1/\sqrt{2}$

concavity: $f''(x)$ has the same sign as $-2x(3-2x^2)$
$= -4x(\frac{3}{2}-x^2) = -4x(\sqrt{\frac{3}{2}}+x)(\sqrt{\frac{3}{2}}-x)$, so:

up | down | up | down

$-\sqrt{3/2}$ | $x=0$ | $\sqrt{3/2}$

Rough sketch:



(13) (graphing) Sketch the graph of $f(x) = \dfrac{-3x^2+4x}{x^2-4}$.

- Find & classify vertical asymptotes (find limits on both sides)
- Find horizontal asymptotes.

discontinuous at $x^2-4=0$, ie $x=\pm 2$.

$$\lim_{x\to(-2)^-} f(x) = \frac{-20}{0^+} = -\infty \qquad \lim_{x\to 2^-} f(x) = \frac{-4}{0^-} = \infty$$

$$\lim_{x\to(-2)^+} f(x) = \frac{-20}{0^-} = \infty \qquad \lim_{x\to 2^+} f(x) = \frac{-4}{0^+} = -\infty.$$

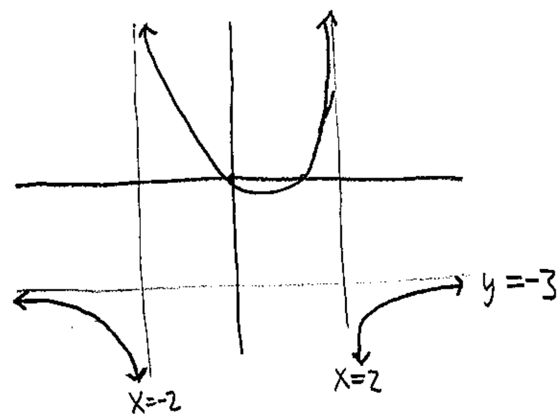
vertical asymptotes at $x=\pm 2$.

horizontal:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{-3+4/x}{1-4/x^2} = \frac{-3+0}{1-0}$$
$$= -3$$

horiz. asymptote at $y=-3$.

rough sketch w/ this information:



(14) (Related rates)

A certain town has a workforce of 50,000 people, of whom 10% are unemployed. Suppose that the workforce is growing at a rate of 900 people per month, and 400 new jobs are being created per month. What is the rate of change of the unemployment rate (expressed in percent per month)?

Define:

$W(t)$ = # people in the workforce

$U(t)$ = portion unemployed (between 0 and 1)

$J(t)$ = # people employed.

These are related by

$$W(t) - W(t)\cdot U(t) = J(t).$$

We are given

$W(0) = 50,000$ and $W'(0) = 900$

$U(0) = 0.1$ $\quad J'(0) = 400$

$\Rightarrow J(0) = 45,000$

Differentiate:

$$W' - W'U - W\cdot U' = J'$$

$$W' - W'U - J' = W\cdot U'$$

$$\frac{W' - W'U - J'}{W} = U'$$

So at $t=0$:

$$U'(0) = \frac{900 - 900\cdot 0.1 - 400}{50,000}$$

$$= \frac{900 - 90 - 400}{50.000}$$

$$= \frac{410}{50,000}$$

$$= \frac{82}{10,000} = \underline{0.0082}$$

so $\boxed{\text{unemployment is increasing at a rate of } 0.82\% \text{ per month.}}$