

Study guide

- Read Saracino, §12.
- **Definitions:** homomorphism, isomorphism, automorphism.

1. Let $\phi : G \rightarrow H$ be a group homomorphism.
 - (a) Prove that if ϕ is injective and H is abelian, then G is abelian.
 - (b) Prove that if ϕ is surjective and G is abelian, then H is abelian.
 - (c) Prove that if ϕ is surjective and G is cyclic, then H is cyclic.
2. Let G be a group, and define a function $\phi : G \rightarrow G$ by $\phi(g) = g^{-1}$.
 - (a) Prove that ϕ is bijective.
 - (b) Prove that if G is abelian, then ϕ is an isomorphism (or an *automorphism*, since it goes from a group to itself).
 - (c) Prove that if G is not abelian, then ϕ is not a homomorphism.
3. Let G be a cyclic group of order n , and k an integer such that $(k, n) = 1$. Prove that the function $\phi : G \rightarrow G$ given by $\phi(g) = g^k$ is an automorphism (see the definition above).

Hint Make use of an earlier homework problem.

4. Let a be an element of a group G . Define a function $\phi_a : G \rightarrow G$ by
$$\phi_a(x) = axa^{-1}.$$
 - (a) Prove that ϕ_a is an automorphism.
 - (b) Consider the set of elements $a \in G$ such that ϕ_a equal to the identity function (i.e. $\phi_a(x) = x$ for all x). What set is this (this set has come up in other contexts, and has a name)?
5. Let V be a vector space. As usual, denote vector addition by $+$ and scalar multiplication by $\cdot$ (or just juxtaposition).
 - (a) Prove that $(V, +)$ is an abelian group. (This mostly just consists of quoting some of the axioms of a vector space).
 - (b) Prove that if $T : V \rightarrow W$ is a linear transformation of vector spaces, then T is also a group homomorphism from $(V, +)$ to $(W, +)$.
 - (c) (Slightly open-ended question; you'll get full points for saying something reasonable; it can just be a hunch or guess. The question is more subtle than it appears on first glance). Do you think that the converse is true? That is, if $\phi : (V, +) \rightarrow (W, +)$ is a group homomorphism, is it necessarily a linear transformation of vector spaces?