

Study guide

- Read: Saracino §9 and §10 through the first paragraph of p. 93. You can omit the last three pages of §10 for now (on the class equation); we may return to that topic later.
- Understand the proof of Theorem 8.4 (no permutation is both odd and even) via counting inversions.
- Definitions: right-coset $H \cdot a$ and left-coset $a \cdot H$.
- The coset criterion, first version: $a \in H \cdot b \iff ab^{-1} \in H$.
- The coset criterion, second version: $H \cdot a = H \cdot b \iff ab^{-1} \in H$.
- The notation $\equiv_H$, and the fact that it is an equivalence relation.
- Why do cosets partition the group G into equal parts?
- The notation $[G : H]$, called the *index* of H in G , and its definition as the number of right-cosets.
- Lagrange's theorem in its usual form: $|H|$ divides $|G|$ if G is finite.
- The element version of Lagrange: $o(g)$ divides $|G|$ if G is finite.
- Know the definition of a *group action*, which is introduced on this problem set (last problem), rather than in class. We will discuss this notion more later.

1. Suppose that G is a group. Show that if H and K are two subgroups of G such that $(|H|, |K|) = 1$, then $H \cap K = \{e\}$.
2. Let p and q be distinct prime numbers, and suppose that G is a group of order pq .
 - (a) Prove that all proper subgroups of G are cyclic (a “proper subgroup” is a subgroup not equal to all of G).
 - (b) Give an example to show that G itself need not be cyclic.
3. Consider the group D_4 (defined with the notation on page 75). Find the right-cosets of the subgroup H in D_4 , for:
 - (a) $H = \langle f \rangle$,
 - (b) $H = \langle f^2 g \rangle$.
4. Let H be a subgroup of a group G . Define a relation $\equiv_H$ on G as follows.

$$x \equiv_H y \iff x^{-1}y \in H$$

(Note that this is not exactly the same as the relation $\equiv_H$ defined in the text and in class, although they are equivalent if G is abelian.)

- (a) Show that $\equiv_H$ is an equivalence relation on G .
 - (b) Show that the equivalence classes under $\equiv_H$ are the left cosets of H in G .
 - (c) Give an example of a group G and subgroup H to demonstrate that $\equiv_H$ is not always the same relation as $\equiv_H$.
5. Let G be a group. Consider the following relation $\sim$ on G : $a \sim b$ means that there exists $x \in G$ such that $a = xbx^{-1}$ (that is, a is *conjugate* to b). Prove that $\sim$ is an equivalence relation on G .
 6. Let G be a group of order 8, and assume that G is not cyclic. Prove that $a^4 = e$ for every $a \in G$.

7. Let G be a group of order p^2 , where p is a prime number. Prove that G must have a subgroup of order p .
8. Saracino exercise 10.14 (five parts), which shows there are essentially only two order-6 groups.
9. Saracino exercise 10.32, parts (a) and (b) only. This problem defines the important concept of a *group action*, and establishes some basic properties.