Study guide

- Read Saracino, §3 − 5.
- Know how to prove the fundamental theorems from §3: uniqueness of identity and inverse, formulas for inverse of a product or of an inverse, and the quicker criterion to show $h = g^{-1}$.
- Know the definition of $(\mathbb{Z}_n, \oplus)$ and its basic properties.
- Defintions: abelian group, subgroup, $\langle g \rangle$, cyclic group, generator
- Suggested (optional) problems (answers to starred problems are in the back of the book); note that some of these concern material that may not be covered in class until early in the second week:
  - §3: 1*, 7
  - §4: 1*, 4*, 10*
  - §5: 1(abcdef)*, 4*

1. Let G be a group. Prove that G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. (This is why it is so important to remember to switch the order in the formula $(xy)^{-1} = y^{-1}x^{-1}$!)

2. Suppose that we take a regular tetrahedron made out of some rigid material, and label its four corners $1, 2, 3, 4$. Call a permutation $f \in S_4$ a *rigid symmetry* of the tetrahedron if you can physically move the tetrahedron in such a way that corner i moves to corner $f(i)$ for all $i \in \{1, 2, 3, 4\}$. List all rigid symmetries of the tetrahedron, and express each one in disjoint cycle notation.

   Hint You will find 12 rigid symmetries total. If you haven't found them all, try multiplying or inverting the permutations you have found.

3. Prove that the symmetric group S_n is nonabelian if $n \geq 3$.

4. Consider the binary operation $a * b = a + b - 2$ on $\mathbb{Z}$.
   (a) Prove that $(\mathbb{Z}, *)$ is a group.
   (b) Is $(\mathbb{Z}, *)$ a *cyclic* group? If so, give a generator.

5. Suppose that x is an element of a group G. Recall from class that by definition $\langle x \rangle = \left\{ x^k \mid k \in \mathbb{Z} \right\}$.
   (a) Prove that if x has finite order in G, then in fact $\langle x \rangle = \left\{ x^k \mid k \geq 0 \right\}$ (that is, we can list $\langle x \rangle$ by listing only *nonnegative* powers).
   (b) Prove conversely that if x has infinite order, then $\langle x \rangle$ contains elements that are not equal to any nonnegative power of x.

6. Suppose that H is a nonempty subset of a group G, and that for all $x, y \in H$, the element xy^{-1} is in H. Prove that H is a subgroup of G. (This is sometimes a convenient one-step criterion for checking if a subset is a subgroup.)

7. If G is a any group, define the *center* of G, denoted $Z(G)$, to be the set of elements that commute with all elements of G. That is,

$$Z(G) = \{z \in G \mid zx = xz \text{ for all } x \in G\}.$$

   (a) Prove that $Z(G) \leq G$ (that is, $Z(G)$ is a subgroup of G).

(b) Determine $Z(D_4)$, the center of the dihedral group of a square.

(c) Determine $Z(S_3)$, the center of the symmetric group on 3 elements.

> **Note** The remaining problems rely on the notion of *the order of an element* in a group, which will be defined on Tuesday 9/15. You can either save the problems until then or look up the definition in the book.

8. In the group $(\mathbb{Z}_{42}, \oplus)$, find the orders of the elements $1, 2, 3, 4, 5, 6$ and 7.

9. Let $G = \langle x \rangle$ be a cyclic group of order 24. List all elements of G that have order 3.

10. Let G be a group, and $a \in G$. Call a second element $b \in G$ *conjugate to a* if there exists an element $x \in G$ such that $b = xax^{-1}$. Show that if b is conjugate to a, then b has the same order as a.