Study guide

- Read Saracino, the rest of §16.
- Review how to solve quadratic equations by factoring, as in high school algebra. How much of the argument works in an arbitrary ring? What axioms and/or assumptions are needed?
- What ring axioms are needed to proof "negative times negative equals positive?" What is the ring version of that statement?
- Basic properties of rings (Theorem 16.1).
- What does it mean to multiply a ring element by an integer (repeated addition)? What algebraic laws does it obey?
- Notation conventions: what exactly is meant by $0_R, 1_R, -a, a-b, a^{-1}$ in a ring?
- Ring attributes and terms: ring with 1, commutative ring, nontrivial ring, units, unit group $R^\times$.
- Examples of rings: trivial ring $\{0\}$, integers $\mathbb{Z}$, rationals $\mathbb{Q}$, real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, even numbers $2\mathbb{Z}$, quaternions $\mathbb{H}$, 2×2 matrices $\mathrm{Mat}(2,\mathbb{R})$. Which of these rings possess which of the attributes above?

1. Prove the following fact, claimed in class: if R is a nontrivial ring with unity, then $1_R \neq 0_R$. Therefore a ring with unity is nontrivial *if and only if* $1_R \neq 0_R$.

2. (a) List the elements of the unit group $\mathbb{Z}_{20}^\times$ of the ring $\mathbb{Z}_{20}$.
   (b) Find the inverse and the order of each element in $\mathbb{Z}_{20}^\times$.
   (c) (Bonus, for extra credit) Prove that $\mathbb{Z}_{20}^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

3. Let F be a field. Use the notation $\frac{a}{b}$ to denote ab^{-1}, for any $a, b \in F$ with $b \neq 0_F$. Prove that for any $a, b, c, d \in F$ with $b, d \neq 0_F$, the usual addition formula for fractions holds:
$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

4. Suppose that F is a field with four elements, labeled $\{0, 1, a, b\}$ (here $0 = 0_F$ is the additive identity and $1 = 1_F$ is the multiplicative identity). Fill out the following charts to determine the addition table and multiplication table for F. In your submission, it is not necessary to write a full formal proof that your answer is the only way to do it, but briefly summarize (in a paragraph or so) how you found the tables.

$+$	0	1	a	b
0				
1				
a				
b				

$\cdot$	0	1	a	b
0				
1				
a				
b				

   Suggestion: There are many ways to proceed, so just try things out and don't be afraid of a little trial and error. A good way to start is to fill out as much as you can using basic properties of 1 and 0. After that, remember that $F^\times$ is a group, and note that the distributive property will force your hand in a number of places.

5. Let R be a nontrivial ring with unity.

(a) Prove that there are at least two distinct elements $x \in R$ satisfying the equation

$$x^2 = 5x - 6 \cdot 1_R.$$

(b) Prove that if R is a *field*, then there are exactly two distinct solutions to the equation.

(c) Give an example of a nontrivial ring with unity in which there are at least three solutions to this equation, and give three such solutions.

6. Let R be a ring with 1, and let n be a positive integer such that $n1 = 0$ (remember that $n1$ means $1 + 1 + \cdots + 1$ with n summands, and we use the abbreviations 1 and 0 for 1_R and 0_R, since R is clear from context). Prove that for any $a \in R$, we have $na = 0$.