Study guide

- **Read:** Saracino, §0, 1, 2 (Sets and induction; binary operations; groups).
- Review proofs by induction, and proofs about sets such as double-containment proofs. Fluency with these techniques will be assumed throughout the semester.
- **Definitions:** binary operation, associative, commutative, group.
- Know the definition of the general linear group $\mathrm{GL}(n,\mathbb{R})$, and why it is a group.
- Know the definition of the symmetric group S_n, how to multiply its elements in two-line notation, and how to write its elements in disjoint cycle notation. The book does not cover this until §8, and you do not need to know all of that chapter yet; focus on the discussion in class, or read the first two pages of §8.
- The dihedral group D_n (especially D_4). Focus on the presentation in class, which is not in the book in the same form.

Problems on Sets and Induction These problems concern material that we did not explicitly cover in class. Some of it may be new to you; if so, read §0 carefully, and do not be shy to come and ask for help at any of the help hours (see the course website for the up-to-date schedule).

1. Let A, B, C be sets. Prove that
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$
(This is sometimes phrased: "$\cup$ distributes over $\cap$.")

2. Suppose that x is a nonzero real number such that $x + \frac{1}{x}$ is an integer. Prove by induction that $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$.

3. Define the *Fibonacci sequence* f_1, f_2, f_3 as follows:
$$f_1 = f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5,\ f_6 = 8, \cdots$$
and in general
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3.$$
Prove that for all $n \geq 1$, $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$. (*Hint:* use induction.)

Problems on binary operations and groups

4. Suppose that $(G, *)$ is a group, and $x, y \in G$ are elements of G such that $x * y = x$. Prove that $y = e$, the identity element.

5. Give an example of a set S and binary operation $*$ such that $s*(s*s)$ isn't always equal to $(s*s)*s$. (*Suggestion:* you can either write down a formula, or write down an explicit table for a finite set, such as in exercises 1.9, 2.5, and 2.6).

6. Let $D_4 = \{e, f, f^2, f^3, g, fg, f^2g, f^3g\}$ be the "dihedral group" of symmetries of the square, following the notation used in class. Complete the multiplication table for D_4 begun in class.

7. (Relativistic addition) Consider the following operation, on the set $S = (-1, 1)$:
$$a * b = \frac{a+b}{1+ab}.$$

(a) Verify that $*$ is a binary operation on S.

(b) Explain why $*$ is *not* a binary operation on $[-1,1]$.

(c) Verify that $(S,*)$ is a group.

Note: This operation is sometimes called "relativistic addition." It tells how to add velocities, expressed as fractions of the speed of light, in special relativity. The fact that $a*b \in (-1,1)$ means that, in special relativity, two velocities cannot "add" to more than the speed of light, even if they themselves are very close to it.

8. Denote by $\mathrm{SL}(2,\mathbb{R})$ the set of 2×2 matrices A with real entries such that $\det A = 1$. Prove that $(SL(2,\mathbb{R}),\cdot)$ is a group (here, $\cdot$ denotes matrix multiplication). (This is called the "special linear group." It is related to the group $\mathrm{GL}(2,\mathbb{R})$ that we considered in class, which is called the "general linear group.")

9. Suppose that $(G,*)$ is a group, and that $x*x = e$ for all $x \in G$ (here e is the identity element). Prove that $(G,*)$ is abelian (that is, that $*$ is commutative).

Note The expression $x*x$ will usually be written x^2 going forward. Sometimes we write x^{*2} to emphasize the operation being used.

10. (Not handed in) Practice the following two mechanical skills:

(a) Multiplying two elements of the symmetric group S_n written in two-line notation (e.g. as in Exercise 8.1, page 76).

(b) Converting an element of S_n from two-line notation to disjoint cycle notation (e.g. as in Exercise 8.2).

You may use the following webpage to practice random examples until you are confident.

`https://npflueger.com/teaching/350-26fall/drills/Sn-arithmetic.html`

Next week's quiz will include one multiplication in S_n, and one disjoint cycle decomposition, exactly as presented in this drill.

More practice: Not to hand in. Answers to starred problems are in the back of the book. These problems are recommended as good ways to continue to build fluency with this week's material, and I will sometimes use them as exam problems.

- §0: Exercises 7, 15
- §1: Exercises 3*, 6*, 9*
- §2: Exercises 1*, 5*