1. (**4.3.3**) Let $G = D_8$, $H = \langle b \rangle$, and $K = \langle a^2 \rangle$. Find $\mathbf{N}_G(H)$ and $\mathbf{N}_G(K)$.

2. (**5.2.7**) Let G be a group of order 338. Assume x and y are two distinct elements of order two in G. Let $H = \langle x, y \rangle$. What are the possibilities for $|H|$? Can x and y commute? Give your reasons.

3. (**5.2.8**) Let G be a group of order $143 = 11 \times 13$, and, as usual, let $\mathbf{Z}(G)$ denote the center of G. Assume that we have found an element $x \in \mathbf{Z}(G)$ with $x \neq e$. What are the possibilities for the $|\mathbf{Z}(G)|$? Prove any assertions you make.

4. (***5.2.13***) **Proof of Theorem 5.24a.** Let $U, V \leq G$ with $|G : U| < \infty$. Show that $|V : V \cap U| \leq |G : U|$.

5. Suppose that G is a group, and N, H are subgroups with $N \triangleleft G$. Prove that the set

$$NH = \{nh : \ n \in N, h \in H\}$$

is a subgroup of G.

6. Suppose that G is a group of order n, and k is an integer such that $gcd(k, n) = 1$.

   (a) Prove that there exists an integer ℓ such that for all $g \in G$, $(g^k)^\ell = g$.
   Hint: First show that $g^{k \cdot \ell} = g^{k \cdot_n \ell}$, where $\cdot_n$ denotes multiplication modulo n.

   (b) Deduce from (a) that the function $f : \ G \to G$ defined by $f(g) = g^k$ is bijective, and give an explicit description of its inverse function.

   (c) Deduce that if G is a finite group of odd order, then every element of G has a unique "square root" (i.e. for every $g \in G$, there exists a unique $h \in G$ such that $h^2 = g$).

   Note: One way to phrase this fact is that it is possible to compute unique "kth roots" in a group G, assuming that k is relatively prime to $|G|$. This fact has an important application in the design of the commonly-used RSA cryptosystem.

7. (**6.1.1**) Let G be a group of order 12, and Ω a set with 5 elements. Assume that G acts on Ω. Can the action be transitive? Either give an example, or prove that it is impossible.

8. (**6.1.4**) Recall Definition 4.14 of the conjugation action of a group on the set of its subgroups as well as Definition 4.24 of normalizers. Let $G = D_8$ act on the set of all of its subgroups by conjugation.

   (a) Let $H = \langle b \rangle$. In Problem 1, you found $\mathbf{N}_G(H)$. Use the FCP to find the size of the orbit of H. What are the elements in the orbit of H.

   (b) Let $K = \langle a \rangle$. Find $\mathbf{N}_G(K)$, and the orbit of K.

9. (**6.3.4**) A group P acts on a set Ω. We know that $|P| = 81$, and $|\Omega| = 98$. Let Ω_0 be the set of elements of Ω that are fixed by every element of P. In other words,

$$\Omega_0 = \{\alpha \in \Omega \mid g \cdot \alpha = \alpha \text{ for all } g \in P\}.$$

Show $|\Omega_0| = 3k + 2$ for some integer k with $0 \leq k \leq 32$.

Some other good problems to try for additional practice (but not to hand in):
4.1.4, 5.1.6, 5.1.10, 5.1.2, 5.2.5