1. (**2.4.5**) Let G be an infinite cyclic group. Prove that $G \cong (\mathbb{Z}, +)$.

2. (**2.4.7**) Let $G = \mathrm{GL}(2, \mathbb{C})$ be the group of invertible 2×2 matrices with entries in the complex numbers $\mathbb{C}$.
   Let $H = \langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$ be a group consisting of some of the elements of G. What is the order of H? Is H isomorphic to D_8?

3. (**2.4.14**) Let G be abelian, H a group, and $\phi : G \to H$ an onto homomorphism. Does H have to be abelian?

4. (***2.6.4***) **Intersection of two subgroups.** Let G be a group and let H and K be subgroups of G. Show that $H \cap K$ is a subgroup of G.

5. (**2.6.5**) Let G be a group, and let x be an element of G that is *not* in the center. As usual, let $\mathbf{Z}(G)$ and $\mathbf{C}_G(x)$, respectively, denote the center of the group and the centralizer of x in G. Show that
   $$\mathbf{Z}(G) < \mathbf{C}_G(x) < G.$$
   In other words, $\mathbf{Z}(G)$ is a *proper* subgroup of $\mathbf{C}_G(x)$ which in turn is a *proper* subgroup of G.

6. (**2.6.17**) Let $G = \mathrm{SL}(2, p)$ be the group of 2×2 matrices with determinant 1 and with entries in $\mathbb{Z}/p\mathbb{Z}$. Let
   $$H = \{ \begin{bmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{bmatrix} \mid \lambda, \mu \in \mathbb{Z}/p\mathbb{Z}, \lambda \neq 0 \}.$$
   (a) Show that H is a subgroup of G. (Note that λ^{-1} is the multiplicative inverse of λ in $\mathbb{Z}/p\mathbb{Z}$.)
   (b) In the case of $p = 3$, find a familiar group that is isomorphic to H.

7. (**2.6.19**) **Kernels of Homomorphisms.** Let G and H be groups, and $\theta : G \longrightarrow H$ be a homomorphism. The set $\{x \in G \mid \theta(x) = e\}$ is called the *kernel* of θ and denoted by $\ker(\theta)$. Show that $\ker(\theta)$ is a subgroup of G.

8. (***2.6.27***) **Conjugate Subgroups.** Let G be a group, $H \leq G$, and $x \in G$. We use the notation xHx^{-1} to denote the set of elements $\{xhx^{-1} \mid h \in H\}$. ($xHx^{-1}$ is called a *conjugate* of H.)
   (a) Prove that xHx^{-1} is a subgroup of G.
   (b) If H is finite, then how are $|H|$ and $|xHx^{-1}|$ related?
   (c) Prove that H is isomorphic to xHx^{-1}.

9. Suppose that G is a group, and H, K are two subgroups of G. Define the following map $\phi : H \times K \to G$:
$$\phi((h,k)) = hk.$$
   (a) Prove that if G is abelian, then ϕ is a homomorphism. Also give an example of a (nonabelian) group G and subgrops H, K for which ϕ is *not* a homomorphism.
   (b) Prove that ϕ is injective if and only if $H \cap K = \{e\}$.
   (c) Prove that if G is finite and abelian, $H \cap K = \{e\}$, and $|H| \cdot |K| = |G|$, then $G \cong H \times K$.

   (Part (c) provides a crucial tool in the classification of finite abelian groups.)

10. (**Bonus**; worth a small amount of extra credit). Consider the following subgroup of $\mathrm{GL}(2,\mathbb{R})$.
$$H = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$
   Prove that $H = \{A \in M_{2\times 2}(\mathbb{Z}) : \ \det A = 1\}$. (This subgroup is usually called $\mathrm{SL}(2,\mathbb{Z})$.)

Some other good problems to try for additional practice (but not to hand in):
2.4.16, 2.5.11, 2.6.2, 2.6.3, 2,6.9, 2.6.24