



*Amherst College*  
*Department of Mathematics and Statistics*

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MATH 350-01

MIDTERM 2 PRACTICE

FALL 2018

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NAME: Solutions

**Read This First!**

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

**Grading - For Instructor Use Only**

Question:	1	2	3	4	5	Total
Points:	8	8	8	8	8	40
Score:						

This page intentionally left blank. You may use it for scratchwork.

1. [8 points] Let  $G, H$  be two groups. Prove that  $G \times H$  is isomorphic to  $H \times G$ .

Define  $\varphi: G \times H \rightarrow H \times G$  by

$$\varphi(g, h) = (h, g).$$

Observe that

$$\begin{aligned} \varphi((g_1, h_1)(g_2, h_2)) &= \varphi(g_1 g_2, h_1 h_2) \\ &= (h_1 h_2, g_1 g_2) \\ &= (h_1, g_1)(h_2, g_2) \\ &= \varphi(g_1, h_1)\varphi(g_2, h_2). \end{aligned}$$

So  $\varphi$  is a (group) homomorphism.

$\varphi$  is surjective, since  $\forall (h, g) \in H \times G, (h, g) = \varphi(g, h)$ .

$\varphi$  is injective, since

$$\begin{aligned} \varphi(g_1, h_1) = \varphi(g_2, h_2) &\Leftrightarrow (h_1, g_1) = (h_2, g_2) \\ &\Leftrightarrow h_1 = h_2 \text{ \& } g_1 = g_2 \\ &\Leftrightarrow (g_1, h_1) = (g_2, h_2). \end{aligned}$$

So  $\varphi$  is an isomorphism, and  $G \times H \cong H \times G$ .

2. [8 points] Prove that if  $G$  is a cyclic group, then there exists a surjective group homomorphism  $\phi: \mathbb{Z} \rightarrow G$ .

Let  $g$  be a generator of  $G$ .

Define  $\varepsilon: \mathbb{Z} \rightarrow G$  by  $\varepsilon(n) = g^n$ .

This is a group hom since  $\forall m, n \in \mathbb{Z}$ ,

$$\varepsilon(m+n) = g^{m+n} = g^m g^n = \varepsilon(m) \varepsilon(n).$$

$\varepsilon$  is surjective since  $\forall g' \in G$ ,  $\exists n \in \mathbb{Z}$  w/  $g' = g^n$   
(def'n of "generator"),

$$\text{so } g' = \varepsilon(n) \in \text{im } \varepsilon.$$

3. [8 points] ] Let  $R$  be a ring, and  $a \in R$  an element.

(a) Prove that if  $a$  is not a zero-divisor, and  $b, c \in R$  satisfy  $ab = ac$ , then  $b = c$ .

If  $ab = ac$ , then

$$\begin{aligned} ab - ac &= ac - ac \\ &= 0_R \end{aligned}$$

$$\Rightarrow a(b - c) = 0_R.$$

Since  $a$  isn't a ZD, it follows that  $b - c = 0_R$ ,

hence  $b - c + c = 0_R + c$

$$\Rightarrow \underline{b = c},$$

as desired.

(b) Prove that if  $a$  is a zero-divisor, then there exist two elements  $b, c \in R$  with  $b \neq c$  but  $ab = ac$ .

Since  $a$  is a ZD,  $\exists b \in R$  st.  $b \neq 0_R$  &  $ab = 0_R$

Let  $c = 0_R$ .

Then  $ab = 0_R = a \cdot 0_R = ac$ ,

but  $b \neq c$  since we assumed that

$$b \neq 0_R.$$

4. [8 points] Suppose that  $G$  is an abelian group, and let  $H$  be the set of all elements of  $G$  with finite order.

(a) Prove that  $H$  is a normal subgroup of  $G$ .

Subgroup:

closed under mult: if  $a, b \in H$ , then  $\exists m, n \in \mathbb{Z}^+$  w/

$$a^m = b^n = e_G. \text{ Hence since } G \text{ is abelian,}$$

$$(ab)^{mn} = a^{mn} b^{mn} = (e_G)^n (e_G)^m = e_G,$$

$$\text{so } o(ab) \mid mn \Rightarrow o(ab) < \infty \Rightarrow ab \in H.$$

closure under inverse:

$$\text{if } a \in H, \text{ then } \exists n \in \mathbb{Z}^+ \text{ st. } a^n = e_G.$$

$$\text{Then } (a^{-1})^n = (a^n)^{-1} = e_G^{-1} = e_G.$$

So  $a^{-1} \in H$  as well.

Normal:

if  $a \in H$  &  $g \in G$ , then  $\exists n \in \mathbb{Z}^+$  st.  $a^n = e_G$ .

$$\text{So } (gag^{-1})^n = \underbrace{gag^{-1}gag^{-1}g \dots gag^{-1}}_{=e_G} = ga^n g^{-1} = ge_G g^{-1} = e_G.$$

So  $gag^{-1} \in H$  as well.

$\Rightarrow H$  is normal in  $G$ .

(b) Prove that all elements of  $G/H$  besides the identity have infinite order.

Consider any element  $Hg \in G/H$ , and suppose it has finite order. Then  $\exists n \in \mathbb{Z}^+$  st.

$$(Hg)^n = e_{G/H} = He_G.$$

$$\Rightarrow Hg^n = He_G$$

$$\Rightarrow g^n \in H$$

$$\Rightarrow \exists m \in \mathbb{Z}^+ \text{ st. } (g^n)^m = e_G$$

$$\Rightarrow g^{nm} = e_G$$

$$\Rightarrow o(g) < \infty.$$

So in fact  $g$  itself is in  $H$ , i.e.  $Hg = He_G$ .

So the only element of  $G/H$  of finite order is the identity element of  $G/H$ .

5. [8 points] Let  $\phi : R \rightarrow S$  be a surjective ring homomorphism.

(a) Define  $\ker \phi$ .

$$\ker \phi = \{x \in R : \phi(x) = 0_S\}.$$

(b) Prove that  $\ker \phi$  is an ideal of  $R$ .

nonemptiness:  $\phi(0_R) = 0_S$ , so  $0_R \in \ker \phi$ . So  $\ker \phi \neq \emptyset$ .

closure under  $-$ : if  $x, y \in \ker \phi$ , then

$$\phi(x-y) = \phi(x) - \phi(y) = 0_S - 0_S = 0_S,$$

so  $x-y \in \ker \phi$ .

stickey property: if  $x \in \ker \phi$  &  $a \in R$ , then

$$\phi(ax) = \phi(a)\phi(x) = \phi(a) \cdot 0_S = 0_S$$

$$\& \phi(xa) = \phi(x)\phi(a) = 0_S \cdot \phi(a) = 0_S,$$

so  $ax$  &  $xa$  are in  $\ker \phi$  as well.

and  $R$  is a comm  
ring w/ unity

(c) Prove that if  $S$  is a field, then  $\ker \phi$  is a maximal ideal of  $R$ .

By the fund. thm. of ring homs., since  $\phi$  is surjective,

$$S \cong R/\ker \phi.$$

We proved in class that if  $R$  is comm w/ unity, that an ideal  $I \subseteq R$  is max'l iff  $R/I$  is a field.

Hence, since  $S$  is a field &  $R/\ker \phi \cong S$ ,  $\ker \phi$  must be a max'l ideal.