1. Use Young diagrams to prove that the number of partitions of n with odd, distinct parts (also denoted $q_0(n)$) equals the number of partitions λ of n which are invariant under conjugation, that is, for which $\lambda = \lambda'$.

2. Use generating functions to prove that the number of partitions of n into distinct parts equals the number of partitions of n where each part is odd.

3. Prove that $p_n \leq p_{n-1} + p_{n-2}$ for $n \geq 1$ by considering first the number of partitions of n that have at least two parts equal to 1 , then the other partitions. Then use this to establish that $p_n \leq F_{n+1}$ for $n \geq 0$, where F_k denotes the k th Fibonacci number

4. A *composition* of n is a list of positive integers $\langle a_1, a_2, \ldots, a_k \rangle$ whose sum is n, where the order of the integers matters. For example, there are four different compositions of $n = 3$: $\langle 3 \rangle, \langle 2, 1 \rangle, \langle 1, 2 \rangle$, and $\langle 1, 1, 1 \rangle$. Let c_n denote the number of compositions of n, and let $c_{n,k}$ denote the number of compositions of n into exactly k parts.

   (a) Compute the value of $c_{n,k}$ for each k and n with $1 \leq k \leq n$ and $1 \leq n \leq 5$ by listing all the compositions, and then calculate the value of c_n for $1 \leq n \leq 5$.

   (b) Using these examples, conjecture formulas for $c_{n,k}$ and c_n, for arbitrary positive integers n and k. Then prove that your formulas are correct.

5. Let λ be a partition of n. A *standard Young tableau* on λ is a labeling of the Young diagram of λ in which a distinct integer from $\{1, 2, \cdots, n\}$ is written in each box, such that each number is less than the number to its left (if there is one) and the number above it (if there is one). For example, there are five standard Young tableaux on the partition $(3, 2)$, which are shown below.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

   (a) Let $T_{a,b}$ denote the number of standard Young tableaux on the partition $\lambda = (a, b)$ (where $a \geq b$). For example, $T_{3,2} = 5$, as shown above. Prove that $T_{a,b}$ is equal to the generalized Catalan number $C_{a,b}$ described on Problem Set 9.

   (b) Let λ be the two-column partition $(2, \cdots, 2, 1, \cdots 1)$, where there are a 2's followed by b 1's. Find a formula for the number of standard Young tableaux on λ in terms of a and b. (Hint: you can do this however you like, but there is a way to do it with almost no computation.)

   (c) Let λ be the "hook" partition $(a + 1, 1, 1, \cdots, 1)$, where there are b 1's. Find a formula for the number of standard Young tableaux on λ.