



*Amherst College*  
*Department of Mathematics and Statistics*

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MATH 272

MIDTERM 2

FALL 2017

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NAME: Solutions

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**Read This First!**

- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. You may use the backs of pages for additional work space.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

**Grading - For Instructor Use Only**

Question:	1	2	3	4	5	6	Total
Points:	9	9	9	9	9	9	54
Score:							

1. **Short answer questions.** No explanations are necessary.

(a) [3 points] Give a basis for each of the following vector spaces.

•  $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

•  $M_{2 \times 2}$  (the space of  $2 \times 2$  matrices)

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

•  $\mathcal{P}_2$  (the space of polynomials of degree at most 2)

$$\{1, X, X^2\}$$

(b) [3 points] Suppose that  $B$  and  $B'$  are two bases for a vector space  $V$ , and the change of basis matrix is

$$[T]_{B'}^B = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

Let  $\vec{v} \in V$  be a vector such that  $[\vec{v}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Determine  $[\vec{v}]_{B'}$ .

$$\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 7 \end{bmatrix}}$$

(c) [3 points] Suppose that  $T: \mathbb{R}^5 \rightarrow \mathbb{R}^3$  is a linear transformation, and that  $T$  is surjective (onto). Determine  $\dim N(T)$  (where  $N(T)$  denotes the null space of  $T$ ).

$$\begin{aligned} \Rightarrow \quad R(T) &= \mathbb{R}^3 \\ \dim N(T) &= \dim \mathbb{R}^5 - \dim \mathbb{R}^3 \\ &= \boxed{2} \end{aligned}$$

2. [9 points] Consider the following three vectors.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(a) Show that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are linearly independent.

row-reduce:

$$\rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 \end{array} \right] \xrightarrow{2-2z, 1-2z}$$

$$\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Pivot in each column  $\Rightarrow$  solns to  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \vec{x} = \vec{0}$  have no free variables  
 $\Rightarrow$  columns are lin. indep.

(b) Find the unique scalars  $c_1, c_2, c_3$  such that the vector

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

is equal to  $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$ .

same row ops., but w/ the aug. matrix:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 3 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\boxed{c_1 = 0, c_2 = -1, c_3 = 3}$$

Note: this was  
self-check problem  
2.3.23.

3. [9 points] Consider the two bases  $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$  and  $B' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$  for  $\mathbb{R}^2$ . Find the change of basis matrix  $[I]_{B'}^B$ .

$$\left[ \begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array} \right] \rightsquigarrow \left[ \begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array} \right] \rightsquigarrow \left[ \begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right]$$

$$\text{so } \left[ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right]_{B'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \end{array} \right] \rightsquigarrow \left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array} \right]$$

$$\text{so } \left[ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$$\boxed{[I]_{B'}^B = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}}$$

4. [9 points] Let  $T: V \rightarrow V$  be a linear transformation.

(a) Prove that  $T(\vec{0}) = \vec{0}$ .

Here are two possible proofs:

proof 1

$$\forall \vec{v} \in V, \quad T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v}) \quad (\text{linearity})$$

but  $0 \cdot \vec{u} = \vec{0}$  for any vector  $\vec{u}$ . So  
this means

$$T(\vec{0}) = \vec{0}.$$

proof 2

Observe that

$$\begin{aligned} T(\vec{0}) &= T(\vec{0} + \vec{0}) \\ &= T(\vec{0}) + T(\vec{0}). \quad (\text{linearity}) \end{aligned}$$

$$\Rightarrow T(\vec{0}) - T(\vec{0}) = T(\vec{0}) + T(\vec{0}) - T(\vec{0})$$

$$\Rightarrow \vec{0} = T(\vec{0}).$$

(b) Suppose that  $\vec{u}, \vec{v}, \vec{w}$  are three vectors in  $V$ , and that the three vectors  $T(\vec{u}), T(\vec{v}), T(\vec{w})$  are linearly independent. Prove that the three vectors  $\vec{u}, \vec{v}, \vec{w}$  are linearly independent.

Suppose that  $c_1, c_2, c_3$  are constants st.

$$c_1 \vec{u} + c_2 \vec{v} + c_3 \vec{w} = \vec{0}.$$

We wish to show that  $c_1 = c_2 = c_3 = 0$ .

Apply  $T$  to both sides:

$$T(c_1 \vec{u} + c_2 \vec{v} + c_3 \vec{w}) = T(\vec{0})$$

$$\Rightarrow \underbrace{c_1 T(\vec{u}) + c_2 T(\vec{v}) + c_3 T(\vec{w})}_{(\text{by linearity})} = \underbrace{\vec{0}}_{(\text{by part a})}.$$

Since  $T(\vec{u}), T(\vec{v}), T(\vec{w})$  are linearly independent,  
it follows that

$$c_1 = c_2 = c_3 = 0.$$

as desired.

5. [9 points] Consider the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) Find a basis for the column space of  $A$ .

row-reduce:

$$\begin{array}{l} - \\ -2x \end{array} \left( \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \right) \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \uparrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots in cols 1 & 3

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\} \text{ is a basis for the column space.}$$

(b) Find a basis for the null space of  $A$ .

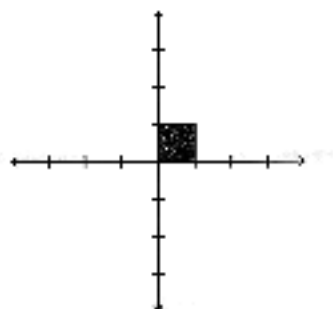
the soln to  $A\vec{x} = \vec{0}$  is given by this aug. matrix in RREF:

$$\left[ \begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_2, x_4 \text{ free} \\ \text{gen'l soln is} \\ x_1 = -2t + s \\ x_2 = t \\ x_3 = -s \\ x_4 = s \end{array}$$

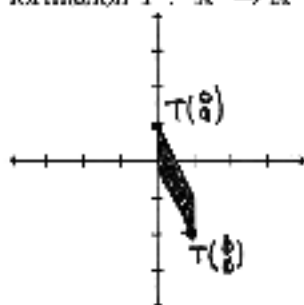
$$\begin{pmatrix} -2t + s \\ t \\ -s \\ s \end{pmatrix} = t \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{so } \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } N(A).$$

6. Consider the square in the plane with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ ,  $(1, 1)$ .



- (a) [3 points] Find the matrix representation (relative to the standard basis) of a linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that transforms the square to the figure shown.

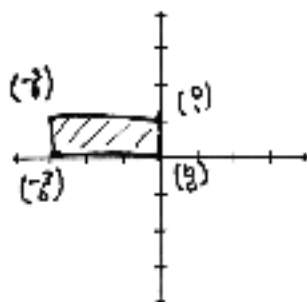


$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \& \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

(vertical shear)

- (b) [3 points] Draw on the axes below the result of transforming the square by the transformation  $S: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  represented (in the standard basis) by  $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$ .



(horiz. dilation)

- (c) [3 points] Determine the matrix representation (in the standard basis) for the transformation  $S \circ T$  (the composition of  $S$  and  $T$ , where  $S$  is the transformation in (b) and  $T$  is the transformation in (a)).

$$\underbrace{\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{trans. in (b)} \\ (S)}} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}_{\substack{\text{trans. in (a)} \\ (T)}} = \boxed{\begin{bmatrix} -3 & 0 \\ -2 & 1 \end{bmatrix}}$$