Amherst College
Department of Mathematics and Statistics

MATH 272 **MIDTERM 1** **FALL 2019**

NAME: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back.
- No calculators or other devices are permitted.
- You may use any of the blank pages to continue answers if you run out of space. Please clearly indicate on the problem's original page if you do so, so that I know to look for it.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	Total
Points:	9	9	9	9	9	45
Score:						

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1. [9 points] Solve the following linear system of equations (your answer should describe all possible solutions, using one or more free variables if necessary).

$$\begin{array}{rrrrrcl} & x_2 & -2x_3 & \cancel{+8x_4} & & = & 5 \\ x_1 & +x_2 & +3x_3 & \cancel{+6x_4} & +\overset{x_4}{\cancel{x_5}} & = & 15 \\ -x_1 & +x_2 & -7x_3 & \cancel{+10x_4} & & = & 2 \end{array}$$

(problem modified from earlier draft)

$$\left(\begin{array}{ccccc|c} 0 & 1 & -2 & \cancel{8} & 0 & 5 \\ 1 & 1 & 3 & \cancel{6} & 1 & 15 \\ -1_{+1} & 1_{+1} & -7_{+3} & \cancel{10}_{\cancel{+6}} & 0_{+1} & 2_{+15} \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R3 \mathrel{+}= R2 \\ R1 \leftrightarrow R2 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 1_{-1} & 3_{+2} & \cancel{6}_{\cancel{-8}} & 1_{-0} & 15_{-5} \\ 0 & 1 & -2 & \cancel{8} & 0 & 5 \\ 0 & 2_{-2} & -4_{+4} & \cancel{16}_{\cancel{-16}} & 1_{-0} & 17_{-10} \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R3 \mathrel{-}= 2R2 \\ R1 \mathrel{-}= R2 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 5 & \cancel{-4} & 1_{-1} & 10_{-7} \\ 0 & 1 & -2 & \cancel{8} & 0 & 5 \\ 0 & 0 & 0 & \cancel{0} & 1 & 7 \end{array}\right)$$

$$\Big\downarrow R1 \mathrel{-}= R3$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 5 & \cancel{-4} & 0 & 3 \\ 0 & 1 & -2 & \cancel{8} & 0 & 5 \\ 0 & 0 & 0 & \cancel{0} & 1 & 7 \end{array}\right)$$

$$\boxed{\begin{array}{l} x_1 = 3 - 5x_3 \cancel{+4x_4} \\ x_2 = 5 + 2x_3 \cancel{-8x_4} \\ x_3, \cancel{x_4} \text{ free} \\ x_4 = 7 \end{array}}$$

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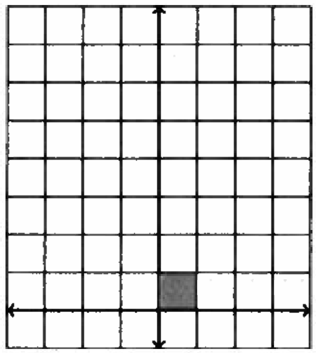
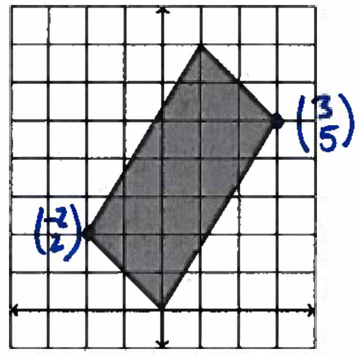
2. [9 points] Suppose that A is a 2×2 matrix that transforms the unit square in the plane in the manner shown below.





(a) Determine A. There are more than one possible answer; you only need to give one.

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ & $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$,

so by "column extraction" $\boxed{A = \begin{pmatrix} 3 & -2 \\ 5 & 2 \end{pmatrix}}$

alt. answer: $A = \begin{pmatrix} -2 & 3 \\ 2 & 5 \end{pmatrix}$.

(b) Using the matrix A you found in part (a), determine the area of the parallelogram in the second picture.

area $= |\det A| = 3 \cdot 2 - (-2) \cdot 5$

$= \boxed{16}$

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3. [9 points] Determine all values of λ for which the following 4×4 matrix is *not* invertible.

$$A = \begin{pmatrix} 0 & \lambda & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & 5 & 7 & 5 \end{pmatrix}$$

evaluating detA by cofactors in 1st row:

$$\det A = 0 - \lambda \cdot \det\begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 7 & 5 \end{pmatrix} + 0 - 1 \cdot \det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ 2 & 5 & 7 \end{pmatrix}$$

(triangular matrix)

3x3 formula

$$= -\lambda \cdot (2 \cdot 3 \cdot 5) - 1 \cdot (2 \cdot 2 \cdot 7 + 1 \cdot 3 \cdot 2 + 0 \cdot 1 \cdot 5 - 2 \cdot 3 \cdot 5 - 1 \cdot 1 \cdot 7 - 0 \cdot 2 \cdot 2)$$

$$= -30\lambda - 1 \cdot (28 + 6 + 0 - 30 - 7 - 0)$$

$$= -30\lambda - 1 \cdot (-3)$$

$$= 3 - 30\lambda.$$

A not inv. $\Leftrightarrow$ $3 - 30\lambda = 0$

$\Leftrightarrow$ $\boxed{\lambda = 1/10}$

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4. [9 points] Describe the set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that can be expressed as linear combinations of the two vectors $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$. Your answer should take the form of an equation in terms of x, y, and z.

$$\left(\begin{array}{cc|c} 2_{-2} & 1_{-10} & x_{-2y} \\ 1 & 5 & y \\ -1_{+1} & 1_{+5} & z_{+y} \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R1 -= 2R2 \\ R3 += R2 \\ R1 \leftrightarrow R2 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 5 & y \\ 0 & -9 & x-2y \\ 0 & 6_{-6} & y+z_{+\frac{2}{3}x-\frac{4}{3}y} \end{array}\right)$$

$$\Big\downarrow R3 += \tfrac{2}{3}R2$$

$$\left(\begin{array}{cc|c} 1 & 5 & y \\ 0 & -9 & x-2y \\ 0 & 0 & \frac{2}{3}x-\frac{1}{3}y+z \end{array}\right)$$

This is inconsistent if $\frac{2}{3}x - \frac{1}{3}y + z \neq 0$.

On the other hand, if $\frac{2}{3}x-\frac{1}{3}y+z=0$, then this matrix is in REF (not RREF, though) & we see that a (unique) sol'n to $c_1\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ exists.

$$\boxed{\tfrac{2}{3}x - \tfrac{1}{3}y + z = 0}$$ (equivalently, $\underline{y = 2x+3z}$).

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5. [9 points] Suppose that A is an $m \times n$ matrix (not necessarily square), $\vec{b}$ is a vector in $\mathbb{R}^m$, and $\vec{u}$ is a vector in $\mathbb{R}^n$ such that $A\vec{u} = \vec{b}$.

(a) Prove that if $\vec{u}$ is the *unique* solution to the matrix equation $A\vec{x} = \vec{b}$, then the homogeneous equation $A\vec{x} = \vec{0}$ has no *nontrivial* solutions.

$\lightning$ Suppose $\vec{u}$ is the unique sol'n to $A\vec{x}=\vec{b}$, and $A\vec{x}=\vec{0}$ has a nontriv. sol'n $\vec{x}=\vec{w}\neq\vec{0}$.

Then $$A(\vec{u}+\vec{w}) = A\vec{u}+A\vec{w} = \vec{b}+\vec{0} = \vec{b}.$$

So $\vec{u}+\vec{w}$ is another sol'n to $A\vec{x}=\vec{b}$. But $\vec{u}+\vec{w}\neq\vec{u}$, contradicting the uniqueness of $\vec{u}$. $\lightning$.

So if $\vec{u}$ is unique, then $A\vec{x}=\vec{0}$ has no nontriv. sol'ns.

(b) Prove, conversely, that if the homogeneous equation $A\vec{x} = \vec{0}$ has no nontrivial solutions, then $\vec{u}$ is the *unique* solution to the (inhomogeneous) matrix equation $A\vec{x} = \vec{b}$.

Suppose that $\vec{v}$ is also a solution to $A\vec{x}=\vec{b}$.

We will prove that $\vec{u}=\vec{v}$.

Observe that $$A(\vec{u}-\vec{v}) = A\vec{u}-A\vec{v} = \vec{b}-\vec{b} = \vec{0}.$$

So $\vec{u}-\vec{v}$ is a sol'n to $A\vec{x}=\vec{0}$. Since this homog. eq'n has no nontrivial sol'ns, $\vec{u}-\vec{v}$ is trivial, i.e. $\vec{u}-\vec{v}=\vec{0}$.

So indeed $\underline{\vec{u}=\vec{v}}$, hence $\vec{u}$ is the <u>unique</u> sol'n to $A\vec{x}=\vec{b}$.

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