Study guide

- (§4.3) Know the following terms: injective (AKA one-to-one), surjective (AKA onto), invertible, isomorphism.
- (§4.3) Understand and be able to prove: a linear transformation T is injective if and only if $\dim N(T) = 0$. (This is theorem 8 in the text.)
- (§4.4) Know the definition of $[T]_B^{B'}$.
- (§4.4) Know how to compute the matrix $[T]_B^{B'}$ one column at a time.
- (§4.4) Know how to convert between matrix representations in different bases.
- (§5.1) Know the terms: eigenvector, eigenvalue, characteristic equation.
- (§5.2) Know how to find the eigenvalues of a matrix, and how to find the eigenspace for each eigenvalue.
- (§5.2) Know how to diagonalize a matrix.

Textbook problems

- §4.4: 3, 13 parts (a) and (b), 20
- §5.1: 8, 16
- §5.2: 20, 22

Supplemental problems:

1. In class, one of you noticed that the phrases "injective" and "surjective" seem to be similar in spirit to the ideas of "span" and "linear independence." The purpose of this exercise is to explain the logical link. Suppose that V is a vector space, and $S = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is a set of vectors in V. Define a transformation $T : \mathbb{R}^n \to V$ by the following formula.

$$T \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

   (a) Prove that T is a linear transformation.

   (b) Prove that S is linearly independent if and only if T is injective.

   (c) Prove that S spans V if and only if T is surjective.

   (d) Prove that S is a basis for V if and only if T is an isomorphism.

2. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection across the line $y = 7x$.

   (a) Find a nonzero vector $\vec{u}$ such that $T(\vec{u}) = \vec{u}$ (i.e. an eigenvector for eigenvalue $\lambda = 1$). You can do this without any computation; think geometrically about T.

   (b) Find a nonzero vector $\vec{v}$ such that $T(\vec{v}) = -\vec{v}$ (in other words, an eigenvector for eigenvalue $\lambda = -1$). Again, you can do this without much computation; it's useful to think about the dot product with $\vec{u}$.

   (c) Let $B = \{\vec{u}, \vec{v}\}$, where these are the vectors you found in parts (a), (b). This is a basis for $\mathbb{R}^2$ (you don't need to prove this). Find $[T]_B$ (this should not require any computations at all; just use the equations $T(\vec{u}) = \vec{u}$ and $T(\vec{v}) = -\vec{v}$).

   (d) Compute $[I]_B^S$, and use it to compute $[T]_S$ (here, S is the standard basis for $\mathbb{R}^2$).

3. In a certain city, researchers aim to model the progression of flu infection. The current state of the infection is recorded in a vector $\begin{pmatrix} h \\ s \end{pmatrix}$, where h denotes the number of healthy people (in millions), and s denotes the number of sick people (in millions). The progression of the disease week-to-week can be modeled by the following equation. Here h', s' denote the number (in millions) of healthy and sick people one week later.

$$\begin{pmatrix} h' \\ s' \end{pmatrix} = \begin{pmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{pmatrix} \begin{pmatrix} h \\ s \end{pmatrix}$$

(a) According to this model, what percentage of healthy people become sick in the following week? What percentage of sick people become healthy?

(b) Determine the eigenvalues of the matrix in the model above, and find a corresponding eigenvector for each eigenvalue.

(c) Suppose that initially, the city has 14 million healthy people and 0 sick people. Express this state as a vector, and write it as a linear combination of the eigenvectors you found in the previous part.

(d) Using your answer in the previous part, obtain a formula for the state (number of healthy people and number of sick people) after n weeks. Your formula should be expressed in terms of nth powers of the eigenvalues you found in (b).

(e) What does your answer in (d) suggest about the long-term state of the disease? In other words, after many weeks have passed, roughly how many people do you expect to be sick in the city?

Note: this problem touches on several ideas about "Markov Chains," which are discussed in detail in §5.4 of our text. You may find it useful to read that section to think about this problem, but it is not necessary to do so (and I will expect you to know any vocabulary or theorems from that section).