F 9/27
class 15
(climate strike day)

Recall: for a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det A = ad-bc$ means two things:

1) The area expansion factor of the tranformation $\mathbb{R}^2 \to \mathbb{R}^2$ represented by A. w/ a ± sign telling whether it reverses orientation;

2) It detects whether A is invertible:

$$A^{-1} \text{ exists} \iff \det A \neq 0.$$

Today, we generalize to all dimensions & state two algorithms.

Informal defn: for an $n \times n$ matrix A, the determinant of A denoted $\det A$, is the factor by which A expands/contracts volume, when viewed as a transformation $\mathbb{R}^n \to \mathbb{R}^n$. It is given a ± sign according to whether it preserves or reverses orientation.

Fact for square A.
A^{-1} exists
$\Updownarrow$
$\det A \neq 0$.

(this is "informal" since I haven't defined "volume" in n dimensions.)

Handy algebraic facts:
1) $\det AB = \det A \cdot \det B$
2) $\det A^t = \det A$
warning
$\det(A+B) \neq \det A + \det B$.

qu How do you compute ~~A~~ $\det A$? (for larger than 2×2?)

$\boxed{\text{method 1}}$ row-reduction + keeping score.

Fact! Suppose B is obtained from A by a row operation. Then:

if the row op. was:	then the determinants satisfy:
$R_i \mathrel{+}= c R_j$	$\det B = \det A$
$R_i \mathrel{*}= c$	$\det B = c \cdot \det A$
$R_i \longleftrightarrow R_j$	$\det B = -\det A$

(side note: "column operations" work the same way!)

Fact 2 $\det I = 1$ (I means "do nothing", so no volume expansion).

geometric reason // just say out loud in class.

$R_i \mathrel{+}= cR_j$ takes all columns of A & "shears" them.

this doesn't change volume.

$R_i \mathrel{*}= c$ magnifies one dim. by c.
This scales volume by $|c|$, & flips orientation if $c<0$.

$R_i \leftrightarrow R_j$ "reflects" all columns across the plane $x_i = x_j$.
This preserves volume, but reverses orientation.

eg.1

$$\det\begin{pmatrix} 0 & 5 & 2 \\ 1 & 3 & 2 \\ 2 & 9 & 6 \end{pmatrix} \overset{(R1\leftrightarrow R2)}{=} -\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 2 \\ 2 & 9 & 6 \end{pmatrix}$$
$$\begin{array}{ccc} -2 & -6 & -4 \end{array}$$

$(R3 \mathrel{-}= 2\cdot R1)$

$$= -\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 2 \\ 0 & 3 & 2 \end{pmatrix}$$
$$\begin{array}{ccc} & -3 & -6/5 \end{array}$$

$(R3 \mathrel{-}= \frac{3}{5} R2)$

$$(*) \quad = -\det\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 2 \\ 0 & 0 & 4/5 \end{pmatrix}$$

a few more += operations clear above the pivots without changing the determinant.

$$= -\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4/5 \end{pmatrix}$$

$\begin{pmatrix} R2 & * = 1/5 \\ R3 & * = 5/4 \end{pmatrix}$

$$= -5 \cdot \frac{4}{5} \cdot \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \boxed{-4} \quad \text{(using } \det I_3 = 1\text{)}.$$

<u>shortcut: "triangular" matrices</u> (to go straight from (*) to answer)

no need to go all the way to RREF if all you need is det A!

$$\det\begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \lambda_1\lambda_2\cdots\lambda_n.$$

<u>eg</u> $\det\begin{pmatrix} 3 & 7 & \pi \\ 0 & 1 & 10^6 \\ 0 & 0 & 2 \end{pmatrix} = 3\cdot1\cdot2 = 6.$

reason: can cancel above pivots w/ $R_i \mathrel{+}= c R_j$ <u>only</u>,
& the scale $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ to I.

Monday.

<u>eg</u>2 $\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \overset{(R2,R3,R4 \mathrel{-}= R1)}{=} \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2_{-2} & 4_{-2} & 8_{-2} \\ 0 & 3_{-3} & 9_{-3} & 27_{-3} \end{pmatrix}$

$$\overset{\substack{R3 \mathrel{-}= 2R2 \\ R4 \mathrel{-}= 3R2}}{=} \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 6_{-6} & 24_{-18} \end{pmatrix}$$

$$\overset{R4 \mathrel{-}= 3R3}{=} \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix} = 1\cdot1\cdot2\cdot6$$

$$= \boxed{12}$$

eg 3 rederivation of 2x2 formula: (when $a \neq 0$)

$$\det\begin{pmatrix} a & b \\ c & d \\ -c & -\frac{c}{a}\cdot b \end{pmatrix} \overset{R2-\frac{c}{a}R1}{=} \begin{pmatrix} a & b \\ 0 & d-\frac{bc}{a} \end{pmatrix} = a\cdot\left(d-\frac{bc}{a}\right)$$
$$= \underline{ad-bc}.$$

$\boxed{\text{Method 2}}$ Cofactor expansion. (recursive... expresses $n\times n$ determinant w/ $(n-1)\times(n-1)$ determinants)

For $n\times n$ matrix A, the cofactor of entry (i,j) is

$$C_{ij} := (-1)^{i+j}\cdot\left[\text{det. of } A \text{ w/ row } i \text{ \& column } j \text{ removed}\right].$$

visualize this as: $\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

Fact for any row R_i, in an $n\times n$ matrix A,

$$\det A = \sum_{j=1}^{n} a_{ij}\cdot C_{ij}. \quad (\text{"expanding along a row"})$$

& for any column j,

$$\det A = \sum_{i=1}^{n} a_{ij}\cdot C_{ij}.$$

eg 1 (again) expanding along row 1:

$$\det\begin{pmatrix} 0 & 5 & 2 \\ 1 & 3 & 2 \\ 2 & 9 & 6 \end{pmatrix} = 0\cdot\det\begin{pmatrix} 3 & 2 \\ 9 & 6 \end{pmatrix} - 5\cdot\det\begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} + 2\cdot\det\begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix}$$

$$= 0 - 5\cdot(1\cdot 6 - 2\cdot 2) + 2\cdot(1\cdot 9 - 3\cdot 2) \quad (\text{2x2 formula})$$

$$= -5\cdot 2 + 2\cdot 3 = \boxed{-4} \quad (\text{as we found before!})$$

M 9/30: begin here.

Notice: 0's are great! You can skip terms:

briefly discussed Friday: finish Monday.

eg. 4

$$\det\begin{pmatrix} 5 & 0 & 0 & 1 \\ 6 & 2 & 17 & 18 \\ 7 & 0 & 1 & 19 \\ 4 & 0 & 0 & 2 \end{pmatrix}$$

expand along column 2:

mind the sign!

$$= -0\cdot\det\begin{pmatrix}\text{don't care!}\end{pmatrix} + 2\cdot\det\begin{pmatrix} 5 & 0 & 1 \\ 7 & 1 & 19 \\ 4 & 0 & 2 \end{pmatrix} - 0\cdot\det\begin{pmatrix}\text{don't care!}\end{pmatrix} + 0\det(\sim)$$

$$= 2\cdot\det\begin{pmatrix} 5 & 0 & 1 \\ 7 & 1 & 19 \\ 4 & 0 & 2 \end{pmatrix}$$

$$= 2\cdot\left[-0 + 1\cdot\det\begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix} - 0\right]$$

$$= 2\cdot(5\cdot 2 - 1\cdot 4) = \boxed{12}$$

qu which method is better?

ans It depends!

- Cofactors are great when a row or column has lots of 0's, and works quickly for 3x3 matrices.
- Row-reduction is better (usually) for bigger matrices, or when you see an operation that simplifies things.
- Sometimes, a hybrid approach is best. (do a row op, then expand).

R2 -= 2R1
R3 -= 3R1
R4 -= 4R1

eg.
$$\det\begin{pmatrix} 1 & 7 & 2 & 5 \\ 2 & 16 & 4 & 10 \\ 3 & 4 & 11 & 15 \\ 4 & 40 & 78 & 27 \end{pmatrix} = \det\begin{pmatrix} 1 & 7 & 2 & 5 \\ 0 & 2 & 0 & 0 \\ 0 & -17 & 5 & 0 \\ 0 & 12 & 70 & 7 \end{pmatrix}$$

(expand col. 1)
$$= 1\cdot\det\begin{pmatrix} 2 & 0 & 0 \\ -17 & 5 & 0 \\ 12 & 70 & 7 \end{pmatrix} - 0 + 0 - 0$$

$$= 1\cdot(2\cdot 5\cdot 7) = \boxed{70}$$