

①

a)

$\mathbb{R}^3$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$M_{2 \times 2}$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$P_2$

$$\{1, x, x^2\}$$

(many other answers are possible, of course).

b)  $[\tilde{v}]_{\mathcal{B}}$  =  $\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 7 \end{pmatrix}}$

c)  $\text{proj}_{(1)} \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$

$$= \frac{(1) \cdot \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}}{(1) \cdot (1)} \cdot (1)$$
$$= \frac{2+3+7}{1+1+1} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}}$$

②

$$B = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

$$B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$$

$$\left[ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right]_{B'} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{by solving: } \begin{pmatrix} 1 & -1 & | & 3 \\ 1 & 1 & | & 1 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & -1 & | & 3 \\ 0 & -2 & | & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \end{pmatrix}$$

$$\& \left[ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right]_{B'} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{by solving: } \begin{pmatrix} 1 & -1 & | & 1 \\ 1 & 1 & | & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & | & 1 \\ 0 & -2 & | & 2 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & -1 \end{pmatrix}$$

$$\text{Hence } [I]_{B'}^B = \boxed{\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}}$$

③ a)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{\substack{R_2 \rightarrow 2R_1 \\ R_4 \rightarrow 2R_1}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{R_2 \leftrightarrow R_3 \\ R_1 \rightarrow 2R_2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{RREF})$$

Pivots in columns 1 & 2, so  $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 4 \end{pmatrix} \right\}$  form a basis of  $W$ .  
(i.e.  $\{\vec{u}, \vec{v}\}$ ).

b)  $\dim W = 2$  (two elements in the basis found in (a)).

c) Gram-Schmidt applied to  $\vec{u}, \vec{v}$ :

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{2+8+0+8}{1+4+0+4} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 4 \end{pmatrix}$$

$$\vec{v} - \text{proj}_{\vec{u}}(\vec{v}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

so  $\left\{ \vec{u}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$  form an orthogonal basis.

For an orthonormal basis, normalize both:

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{1+4+0+4}} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix}$$

$$\frac{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(other answer possible, of course).

d) There are two approaches here that we've discussed in this class (& many others using other techniques).

method 1: least-squares:

$W$  has basis  $\vec{u}, \vec{v}$ ,  
so we can optimize

$$\|c_1 \vec{u} + c_2 \vec{v} - \vec{b}\|$$

by solving

$$\begin{pmatrix} \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{v} \\ \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{v} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{u} \cdot \vec{b} \\ \vec{v} \cdot \vec{b} \end{pmatrix}$$

$$\begin{pmatrix} 9 & 18 \\ 18 & 37 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

$$\left( \begin{array}{cc|c} 9 & 18 & 5 \\ 18 & 37 & 11 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 9 & 18 & 5 \\ 0 & 1 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{cc|c} 9 & 0 & -13 \\ 0 & 1 & 1 \end{array} \right) \rightarrow$$

so  $c_1 = -\frac{13}{9}$ ,  $c_2 = 1$  is optimal. The closest point is

$$-\frac{13}{9} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 4 \\ 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 5/9 \\ 10/9 \\ 1 \\ 10/9 \end{pmatrix}$$

method 2 The special-purpose approximation method for orthogonal sets (see notes on approx in inner product spaces):

Use the orthonormal basis from (c).

Denote the vectors by  $\vec{w}_1, \vec{w}_2$  for convenience:

$$\vec{w}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The nearest combination to  $\vec{b}$  is

$$\text{proj}_{\vec{w}_1}(\vec{b}) + \text{proj}_{\vec{w}_2}(\vec{b})$$

$$= \frac{\vec{w}_1 \cdot \vec{b}}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 + \frac{\vec{w}_2 \cdot \vec{b}}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2$$

$$= \frac{\frac{1}{3} + \frac{2}{3} + 0 + \frac{2}{3}}{1} \vec{w}_1 + \frac{1}{1} \vec{w}_2$$

$$= \frac{5}{3} \cdot \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5/9 \\ 10/9 \\ 1 \\ 10/9 \end{pmatrix}$$

④

Since norms are nonnegative (positive-definiteness), it's equivalent to prove that

$$\|\bar{u} + 2\bar{v}\|^2 > \|\bar{u}\|^2. \quad (\text{square both sides}).$$

Viewing the squared norms as an inner product, observe that

$$\|\bar{u} + 2\bar{v}\|^2 = \langle \bar{u} + 2\bar{v}, \bar{u} + 2\bar{v} \rangle$$

$$= \langle \bar{u} + 2\bar{v}, \bar{u} \rangle + 2 \langle \bar{u} + 2\bar{v}, \bar{v} \rangle \quad (\text{linearity in 2<sup>nd</sup> argument})$$

$$= \langle \bar{u}, \bar{u} \rangle + 2 \langle \bar{v}, \bar{u} \rangle + 2 \langle \bar{u}, \bar{v} \rangle + 4 \langle \bar{v}, \bar{v} \rangle \quad (\text{linearity in 1<sup>st</sup> argument})$$

$$= \langle \bar{u}, \bar{u} \rangle + 4 \langle \bar{v}, \bar{v} \rangle \quad (\text{since } \langle \bar{u}, \bar{v} \rangle = 0, \text{ \& hence } \langle \bar{v}, \bar{u} \rangle = 0 \text{ also, by } \underline{\text{symmetry}})$$

$$= \|\bar{u}\|^2 + 4 \|\bar{v}\|^2$$

$$> \|\bar{u}\|^2 \quad (\text{since } \bar{v} \text{ is nonzero, } \|\bar{v}\| > 0 \text{ by } \underline{\text{positive definiteness}}).$$

So indeed  $\|\bar{u} + 2\bar{v}\|^2 > \|\bar{u}\|^2$ , as desired.

5) a)

$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)dx$  defines an inner product on  $\mathcal{P}[-\pi, \pi]$ .

So the inner products given amount to:

$$\langle f, \sin x \rangle = 7$$

$$\langle f, \cos x \rangle = 13$$

$$\langle \sin x, \sin x \rangle = \pi$$

$$\langle \sin x, \cos x \rangle = 0$$

$$\langle \cos x, \cos x \rangle = \pi.$$

b) Since  $\sin x \perp \cos x$ , the optimum choices are:

$$C_1 = \frac{\langle f, \sin x \rangle}{\langle \sin x, \sin x \rangle} = \frac{7}{\pi} \quad (\text{ensures } (C_1 \sin x + C_2 \cos x - f(x)) \perp \sin x)$$

$$C_2 = \frac{\langle f, \cos x \rangle}{\langle \cos x, \cos x \rangle} = \frac{13}{\pi} \quad (\text{ensures } (C_1 \sin x + C_2 \cos x - f(x)) \perp \cos x).$$

$$\boxed{C_1 = \frac{7}{\pi}, \quad C_2 = \frac{13}{\pi}.}$$

(In other words,  $C_1 \sin x + C_2 \cos x$

is equal to  $\text{proj}_{\sin x} f(x) + \text{proj}_{\cos x} f(x)$ ).

6. [9 points] Suppose that  $A$  is an  $m \times n$  matrix, and  $\vec{u}, \vec{v}, \vec{w}$  are three vectors in  $\mathbb{R}^n$  such that  $\{A\vec{u}, A\vec{v}, A\vec{w}\}$  is a linearly independent set of vectors. Prove that  $\{\vec{u}, \vec{v}, \vec{w}\}$  is also a linearly independent set of vectors.

Suppose that  $c_1, c_2, c_3$  are constants such that

$$c_1 \vec{u} + c_2 \vec{v} + c_3 \vec{w} = \vec{0}.$$

We wish to show that  $c_1 = c_2 = c_3 = 0$ .

Multiply both sides by  $A$ :

$$A(c_1 \vec{u} + c_2 \vec{v} + c_3 \vec{w}) = A \cdot \vec{0}$$

$$\Rightarrow A(c_1 \vec{u}) + A(c_2 \vec{v}) + A(c_3 \vec{w}) = \vec{0}$$

$$\Rightarrow c_1 (A\vec{u}) + c_2 (A\vec{v}) + c_3 (A\vec{w}) = \vec{0}.$$

Since  $\{A\vec{u}, A\vec{v}, A\vec{w}\}$  is linearly independent, it follows that  $c_1 = c_2 = c_3 = 0$ , as desired.

So the only way to write  $\vec{0}$  as a linear combination of  $\vec{u}, \vec{v}, \vec{w}$  is the trivial way, i.e.  $\{\vec{u}, \vec{v}, \vec{w}\}$  is linearly independent.