

①

a)

\mathbb{R}^3

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$M_{2 \times 2}$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

\mathbb{P}_2

$$\{1, x, x^2\}$$

(many other answers are possible, of course).

$$b) [\vec{v}]_{B'} = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 7 \end{pmatrix}}$$

$$c) \dim R(T) = \dim \mathbb{R}^3 - \dim N(T) \\ = 3 - 1 = \boxed{2}$$

②

$$B = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

$$B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$$

$$\left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} \right]_{B'} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{by solving: } \begin{pmatrix} 1 & -1 & | & 3 \\ 1 & -1 & | & 1 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & -1 & | & 3 \\ 0 & -2 & | & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \end{pmatrix}$$

$$\& \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} \right]_{B'} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{by solving: } \begin{pmatrix} 1 & -1 & | & 1 \\ 1 & -1 & | & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & | & 1 \\ 0 & -2 & | & 2 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -1 \end{pmatrix}$$

$$\text{Hence } [I]_{B'}^B = \boxed{\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}}$$

3) a)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{\substack{R_2 \leftarrow 2R_1 \\ R_4 \leftarrow 2R_1}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow{\substack{R_2 \leftrightarrow R_3 \\ R_1 \leftarrow 2R_2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{RREF})$$

Pivots in columns 1 & 2, so $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 4 \end{pmatrix} \right\}$ form a basis of W .
(i.e. $\{\vec{u}, \vec{v}\}$).

b) $\dim W = 2$ (two elements in the basis found in (a)).

c) Gram-Schmidt applied to \vec{u}, \vec{v} :

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{2+8+0+8}{1+4+0+4} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 4/3 \\ 0 \\ 2/3 \end{pmatrix}$$

$$\vec{v} - \text{proj}_{\vec{u}}(\vec{v}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

so $\left\{ \vec{u}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ form an orthogonal basis.

For an orthonormal basis, normalize both:

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{1+4+0+4}} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix}$$

$$\frac{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(other answer possible, of course).

d) There are two approaches here that we've discussed in this class (& many others using other techniques).

method 1 : least-squares:

W has basis \vec{u}, \vec{v} ,
so we can optimize

$$\|c_1\vec{u} + c_2\vec{v} - \vec{b}\|$$

by solving

$$\begin{pmatrix} \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{v} \\ \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{v} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{u} \cdot \vec{b} \\ \vec{v} \cdot \vec{b} \end{pmatrix}$$

$$\begin{pmatrix} 9 & 18 \\ 18 & 37 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 9 & 18 & 5 \\ 18 & 37 & 11 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 9 & 18 & 5 \\ 0 & 1 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{cc|c} 9 & 0 & -13 \\ 0 & 1 & 1 \end{array} \right)$$

so $c_1 = -\frac{13}{9}$, $c_2 = 1$ is optimal. The closest point is

$$-\frac{13}{9} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 4 \\ 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 5/9 \\ 10/9 \\ 1 \\ 10/9 \end{pmatrix}$$

method 2 The special-purpose approximation method for orthogonal sets (see notes on approx. in inner product spaces):

Use the orthonormal basis from (c).

Denote the vectors by \vec{w}_1, \vec{w}_2 for convenience:

$$\vec{w}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix}, \quad \vec{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The nearest combination to \vec{b} is

$$\text{proj}_{\vec{w}_1}(\vec{b}) + \text{proj}_{\vec{w}_2}(\vec{b})$$

$$= \frac{\vec{w}_1 \cdot \vec{b}}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 + \frac{\vec{w}_2 \cdot \vec{b}}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2$$

$$= \frac{\frac{1}{3} + \frac{2}{3} + 0 + \frac{2}{3}}{1} \cdot \vec{w}_1 + \frac{1}{1} \vec{w}_2$$

$$= \frac{5}{3} \cdot \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5/9 \\ 10/9 \\ 1 \\ 10/9 \end{pmatrix}$$

④

Since norms are nonnegative (positive-definiteness), it's equivalent to prove that

$$\|\bar{u} + 2\bar{v}\|^2 > \|\bar{u}\|^2. \quad (\text{square both sides}).$$

Viewing the squared norm as an inner product, observe that

$$\|\bar{u} + 2\bar{v}\|^2 = \langle \bar{u} + 2\bar{v}, \bar{u} + 2\bar{v} \rangle$$

$$= \langle \bar{u} + 2\bar{v}, \bar{u} \rangle + 2 \langle \bar{u} + 2\bar{v}, \bar{v} \rangle \quad (\text{linearity in 2nd argument})$$

$$= \langle \bar{u}, \bar{u} \rangle + 2 \langle \bar{v}, \bar{u} \rangle + 2 \langle \bar{u}, \bar{v} \rangle + 4 \langle \bar{v}, \bar{v} \rangle \quad (\text{linearity in 1st argument})$$

$$= \langle \bar{u}, \bar{u} \rangle + 4 \langle \bar{v}, \bar{v} \rangle \quad (\text{since } \langle \bar{u}, \bar{v} \rangle = 0, \text{ \& hence } \langle \bar{v}, \bar{u} \rangle = 0 \text{ also by } \underline{\text{symmetry}})$$

$$= \|\bar{u}\|^2 + 4 \|\bar{v}\|^2$$

$$> \|\bar{u}\|^2 \quad (\text{since } \bar{v} \text{ is nonzero, } \|\bar{v}\| > 0 \text{ by } \underline{\text{positive definiteness}}).$$

So indeed $\|\bar{u} + 2\bar{v}\|^2 > \|\bar{u}\|^2$ as desired.

5) Row-reducing:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow{\substack{R_3 \leftarrow R_1 \\ R_3 \leftarrow R_2 \\ R_4 \leftarrow 2R_2}} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_2} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) ~~Pivots~~ Pivots in columns 1 & 3, so $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\}$ is a basis for $R(T)$.

(the corresponding columns from the original matrix).

b) using the RREF, the general sol'n to $A\vec{x} = \vec{0}$ can be written

$$\begin{pmatrix} -2x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} \quad (x_2, x_4 \text{ free})$$

$$= x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

so $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $N(T)$.

⑥ a)

$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)dx$ defines an inner product on $\mathcal{C}[-\pi, \pi]$.

So the inner products given amount to:

$$\langle f, \sin x \rangle = 7$$

$$\langle f, \cos x \rangle = 13$$

$$\langle \sin x, \sin x \rangle = \pi$$

$$\langle \sin x, \cos x \rangle = 0$$

$$\langle \cos x, \cos x \rangle = \pi.$$

b) Since $\sin x \perp \cos x$, the optimum choices are:

$$c_1 = \frac{\langle f, \sin x \rangle}{\langle \sin x, \sin x \rangle} = \frac{7}{\pi} \quad (\text{ensures } (c_1 \sin x + c_2 \cos x - f(x)) \perp \sin x)$$

$$c_2 = \frac{\langle f, \cos x \rangle}{\langle \cos x, \cos x \rangle} = \frac{13}{\pi} \quad (\text{ensures } (c_1 \sin x + c_2 \cos x - f(x)) \perp \cos x).$$

$$\boxed{c_1 = \frac{7}{\pi}, \quad c_2 = \frac{13}{\pi}.}$$