

Proofs of claims from the "review of key concepts"
handout

Note: following our in-class convention since §3.3, I'll use $+, \cdot$ instead of \oplus, \odot below.

Claim 1 Any subspace W of a vector space V contains $\vec{0}$.

Pf W is closed under scalar multiplication. So for every $\vec{w} \in W$, $0 \cdot \vec{w} \in W$ as well. But $0 \cdot \vec{v} = \vec{0}$ for all $\vec{v} \in V$.

Claim 2 A list of vectors $\vec{v}_1, \dots, \vec{v}_n \in V$ is linearly independent if and only if

$\forall \vec{v} \in \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$, there are unique coefficients $c_1, \dots, c_n \in \mathbb{R}$ st. $\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$.

Pf 1) Suppose $\vec{v}_1, \dots, \vec{v}_n$ are lin. indep. Suppose also that \vec{v} is in their span. So \exists constants c_1, \dots, c_n st $\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$. To see that they are unique, suppose that

$$\vec{v} = c'_1 \vec{v}_1 + \dots + c'_n \vec{v}_n$$

for some list of constants c'_1, \dots, c'_n ; we will show that in fact $c'_1 = c_1, c'_2 = c_2, \dots, c'_n = c_n$. This will establish that there is no other way to choose the constants.

Indeed: $c'_1 \vec{v}_1 + c'_2 \vec{v}_2 + \dots + c'_n \vec{v}_n = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$

$$\Rightarrow (c'_1 \vec{v}_1 - c_1 \vec{v}_1) + (c'_2 \vec{v}_2 - c_2 \vec{v}_2) + \dots + (c'_n \vec{v}_n - c_n \vec{v}_n) = \vec{0}$$

$$\Rightarrow (c'_1 - c_1) \vec{v}_1 + (c'_2 - c_2) \vec{v}_2 + \dots + (c'_n - c_n) \vec{v}_n = \vec{0}$$

$$\Rightarrow c'_1 - c_1 = c'_2 - c_2 = \dots = c'_n - c_n = 0 \quad (\text{by lin. independence})$$

$$\Rightarrow c'_1 = c_1, c'_2 = c_2, \dots, c'_n = c_n,$$

as claimed.

see
Thm. 9,
pg. 159.

2) For the converse, suppose that all elements of the span have unique coefficients. Then $\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \dots + 0 \cdot \vec{v}_n$ must be the unique way to write $\vec{0}$ as a lin. comb. of $\vec{v}_1, \dots, \vec{v}_n$. So the only way to write

$$\vec{0} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n \quad \text{the list}$$

is if $c_1 = c_2 = \dots = c_n = 0$. Hence $\vec{v}_1, \dots, \vec{v}_n$ is lin. indep.

Claim 3 If $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for V , then $\forall \vec{v} \in V$, there are unique "coordinates" $c_1, \dots, c_n \in \mathbb{R}$ st. $\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$.

Pf ~~Thm 5~~ Since B is a basis, $\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$ and $\vec{v}_1, \dots, \vec{v}_n$ are lin. indep. So by claim 2, all $\vec{v} \in V$ ($= \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$) determine unique constants c_1, \dots, c_n st. $\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$.

see Thm. 5 on pg. 115. Claim 4 $\{\vec{v}_1, \dots, \vec{v}_n\}$ is lin. indep. iff no vector in the list is in the span of the others.

Pf i) Suppose that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is lin. indep. Suppose (for contradiction) that one of them, say \vec{v}_i , is in the span of the others. This means that

$$\vec{v}_i = c_1 \vec{v}_1 + \dots + c_{i-1} \vec{v}_{i-1} + c_{i+1} \vec{v}_{i+1} + \dots + c_n \vec{v}_n.$$

for some constants $c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n$.

This implies that

$$\vec{0} = c_1 \vec{v}_1 + \dots + c_{i-1} \vec{v}_{i-1} + (-1) \vec{v}_i + c_{i+1} \vec{v}_{i+1} + \dots + c_n \vec{v}_n,$$

contradicting the linear independence of $\vec{v}_1, \dots, \vec{v}_n$. \Leftarrow
So none of $\vec{v}_1, \dots, \vec{v}_n$ is a lin. comb. of the others.

2) Conversely, suppose that no \vec{v}_i is a lin. comb. of the others.

↳ Suppose (for contradiction) that $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent. Then $\exists c_1, \dots, c_n \in \mathbb{R}$, not all 0, st

$$\vec{0} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n.$$

Suppose $c_i \neq 0$ (we've assumed at least one of the c_i isn't 0, so choose one). Then:

$$-c_i \vec{v}_i = c_1 \vec{v}_1 + \dots + c_{i-1} \vec{v}_{i-1} + c_{i+1} \vec{v}_{i+1} + \dots + c_n \vec{v}_n$$

$$\begin{aligned} (\text{mult. by } -\frac{1}{c_i}) \Rightarrow \vec{v}_i &= \left(-\frac{c_1}{c_i}\right) \vec{v}_1 + \left(-\frac{c_2}{c_i}\right) \vec{v}_2 \\ &+ \dots + \left(-\frac{c_{i-1}}{c_i}\right) \vec{v}_{i-1} + \left(-\frac{c_{i+1}}{c_i}\right) \vec{v}_{i+1} \\ &+ \dots + \left(-\frac{c_n}{c_i}\right) \vec{v}_n \end{aligned}$$

So \vec{v}_i is a lin. comb. of the others, a contradiction. \Leftarrow

So $\vec{v}_1, \dots, \vec{v}_n$ are in fact lin. indep.

Claim 5 If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is lin. dependent, then

any \vec{v} in their span can be written

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

in infinitely many ways.

Proof There are constants $d_1, \dots, d_n \in \mathbb{R}$, not all 0, st.

$$\vec{0} = d_1 \vec{v}_1 + \dots + d_n \vec{v}_n.$$

Hence whenever \vec{v} can be written

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n,$$

it follows that $\forall t \in \mathbb{R}$,

$$\vec{v} = \vec{v} + t \cdot \vec{0}$$

$$= (c_1 + t \cdot d_1) \vec{v}_1 + (c_2 + t \cdot d_2) \vec{v}_2 + \dots + (c_n + t \cdot d_n) \vec{v}_n.$$

Since not all of d_1, \dots, d_n are 0, the list of coefficients

$$c_1 + t \cdot d_1, \dots, c_n + t \cdot d_n$$

is different for every choice of t (more precisely:

if $d_i \neq 0$, then $c_i + t \cdot d_i = c_i + t' \cdot d_i$ iff

$t = t'$, so the coeff. $c_i + t \cdot d_i$ is different for every value of $t \in \mathbb{R}$).

So \vec{v} is a lin. comb. of $\vec{v}_1, \dots, \vec{v}_n$ in ∞ ways.

Claim 6 $\text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\} = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$
iff $\vec{v}_n \in \text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$.

Proof 1) Suppose $\text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\} = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$.

Then since $\vec{v}_n \in \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$ (it is $0 \cdot \vec{v}_1 + \dots + 0 \cdot \vec{v}_{n-1} + 1 \cdot \vec{v}_n$),

$\vec{v}_n \in \text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$ as well.

2) Suppose that $\vec{v}_n \in \text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$; sees that

$$\vec{v}_n = d_1 \vec{v}_1 + \dots + d_{n-1} \vec{v}_{n-1}.$$

Then $\forall \vec{v} \in \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$,

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n \quad \text{for some } c_1, \dots, c_n \in \mathbb{R},$$

$$\Rightarrow \vec{v} = c_1 \vec{v}_1 + \dots + c_{n-1} \vec{v}_{n-1} + c_n (d_1 \vec{v}_1 + \dots + d_{n-1} \vec{v}_{n-1})$$

$$= (c_1 + d_1 c_n) \vec{v}_1 + (c_2 + d_2 c_n) \vec{v}_2 + \dots + (c_{n-1} + d_{n-1} c_n) \vec{v}_{n-1}$$

$\Rightarrow \vec{v} \in \text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$ as well.

So $\text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\} \subseteq \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$. But also $\text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$

$\subseteq \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$ since $c_1 \vec{v}_1 + \dots + c_{n-1} \vec{v}_{n-1} = c_1 \vec{v}_1 + \dots + c_{n-1} \vec{v}_{n-1} + 0 \cdot \vec{v}_n$.

So $\text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\} = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$, as desired.