1. [12 points] Let a, b, d be positive integers, and suppose that there exist integers u, v such that

$$au + bv = d.$$

Prove that $\gcd(a, b)$ divides d.

2. [12 points] Alice and Bob are using the Elgamal cryptosystem (see the textbook's reference table at the back of the exam packet). They use the following public parameters:

$$p = 41 \qquad\qquad g = 6$$

You will find a multiplication table modulo 41 at the back of the exam packet (you may wish to detach it for easy reference).

Alice chooses

$$a = 19$$

as her private key, and uses this to determine her public key $A = 34$. After Alice publishes this public key, Bob sends her the following ciphertext.

$$c_1 = 12 \qquad\qquad c_2 = 15$$

Determine the plaintext m. **For any modular exponentiation you perform, use a procedure that would scale well to large modulus**, and show enough work to make it clear what procedure you are following. It is possible to do this computation with a fairly small number of multiplications.

3. [12 points] Every day, Alice and Bob perform Diffie-Hellman key exchange, using public parameters p and g (the textbook's Diffie-Hellman reference table is provided at the back of the exam packet). Unfortunately, Alice and Bob do not randomize their secret numbers well.

On Monday, Alice sends Bob the number A, Bob sends Alice the number B, and they establish a shared secret S. On Tuesday, Alice sends A', Bob sends B', and they establish a shared secret S'. Eve examines the numbers A, B, A', B', and discovers the following facts (resulting from poor random number generation).

$$\begin{aligned} g^2 A' &= A \pmod{p} \\ B' &= B^3 \pmod{p} \end{aligned}$$

Show that if Eve manages to learn Monday's shared secret S, then she can quickly determine Tuesday's shared secret S' as well.

More precisely, describe a procedure Eve could follow to efficiently compute the number S'. You may assume that Eve knows p, g, A, B, A', B', and S. Do not assume that Eve knows (or can learn) Alice and Bob's secret numbers a or b. You do not need to write your solution as a program, but be clear about any algorithms Eve will require in her computation, and explain why your method will work.

4. Let p be a prime number, and g be a unit modulo p.

(a) [5 points] Prove that if e, f are positive integers such that $e \equiv f \pmod{p-1}$, then $g^e \equiv g^f \pmod{p}$.

(b) [2 points] Define what it means to say that g is a *primitive root* modulo p.

(c) [5 points] Prove that if g is a primitive root modulo p, then the converse to part (a) is true, namely: if e, f are integers such that $g^e \equiv g^f \pmod{p}$, then $e \equiv f \pmod{p-1}$.

Public parameter creation	
A trusted party chooses and publishes a (large) prime p and an integer g having large prime order in $\mathbb{F}_p^*$.	
Private computations	
Alice	**Bob**
Choose a secret integer a. Compute $A \equiv g^a \pmod{p}$.	Choose a secret integer b. Compute $B \equiv g^b \pmod{p}$.
Public exchange of values	
Alice sends A to Bob	$\longrightarrow$ A
B $\longleftarrow$	Bob sends B to Alice
Further private computations	
Alice	**Bob**
Compute the number $B^a \pmod{p}$.	Compute the number $A^b \pmod{p}$.
The shared secret value is $B^a \equiv (g^b)^a \equiv g^{ab} \equiv (g^a)^b \equiv A^b \pmod{p}$.	

Table 2.2: Diffie–Hellman key exchange

Public parameter creation	
A trusted party chooses and publishes a large prime p and an element g modulo p of large (prime) order.	
Alice	**Bob**
Key creation	
Choose private key $1 \le a \le p-1$. Compute $A = g^a \pmod{p}$. Publish the public key A.	
Encryption	
	Choose plaintext m. Choose random element k. Use Alice's public key A to compute $c_1 = g^k \pmod{p}$ and $c_2 = mA^k \pmod{p}$. Send ciphertext (c_1, c_2) to Alice.
Decryption	
Compute $(c_1^a)^{-1} \cdot c_2 \pmod{p}$. This quantity is equal to m.	

Table 2.3: Elgamal key creation, encryption, and decryption

Multiplication table modulo 41:

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
3	0	3	6	9	12	15	18	21	24	27	30	33	36	39	1	4	7	10	13	16	19
4	0	4	8	12	16	20	24	28	32	36	40	3	7	11	15	19	23	27	31	35	39
5	0	5	10	15	20	25	30	35	40	4	9	14	19	24	29	34	39	3	8	13	18
6	0	6	12	18	24	30	36	1	7	13	19	25	31	37	2	8	14	20	26	32	38
7	0	7	14	21	28	35	1	8	15	22	29	36	2	9	16	23	30	37	3	10	17
8	0	8	16	24	32	40	7	15	23	31	39	6	14	22	30	38	5	13	21	29	37
9	0	9	18	27	36	4	13	22	31	40	8	17	26	35	3	12	21	30	39	7	16
10	0	10	20	30	40	9	19	29	39	8	18	28	38	7	17	27	37	6	16	26	36
11	0	11	22	33	3	14	25	36	6	17	28	39	9	20	31	1	12	23	34	4	15
12	0	12	24	36	7	19	31	2	14	26	38	9	21	33	4	16	28	40	11	23	35
13	0	13	26	39	11	24	37	9	22	35	7	20	33	5	18	31	3	16	29	1	14
14	0	14	28	1	15	29	2	16	30	3	17	31	4	18	32	5	19	33	6	20	34
15	0	15	30	4	19	34	8	23	38	12	27	1	16	31	5	20	35	9	24	39	13
16	0	16	32	7	23	39	14	30	5	21	37	12	28	3	19	35	10	26	1	17	33
17	0	17	34	10	27	3	20	37	13	30	6	23	40	16	33	9	26	2	19	36	12
18	0	18	36	13	31	8	26	3	21	39	16	34	11	29	6	24	1	19	37	14	32
19	0	19	38	16	35	13	32	10	29	7	26	4	23	1	20	39	17	36	14	33	11
20	0	20	40	19	39	18	38	17	37	16	36	15	35	14	34	13	33	12	32	11	31
21	0	21	1	22	2	23	3	24	4	25	5	26	6	27	7	28	8	29	9	30	10
22	0	22	3	25	6	28	9	31	12	34	15	37	18	40	21	2	24	5	27	8	30
23	0	23	5	28	10	33	15	38	20	2	25	7	30	12	35	17	40	22	4	27	9
24	0	24	7	31	14	38	21	4	28	11	35	18	1	25	8	32	15	39	22	5	29
25	0	25	9	34	18	2	27	11	36	20	4	29	13	38	22	6	31	15	40	24	8
26	0	26	11	37	22	7	33	18	3	29	14	40	25	10	36	21	6	32	17	2	28
27	0	27	13	40	26	12	39	25	11	38	24	10	37	23	9	36	22	8	35	21	7
28	0	28	15	2	30	17	4	32	19	6	34	21	8	36	23	10	38	25	12	40	27
29	0	29	17	5	34	22	10	39	27	15	3	32	20	8	37	25	13	1	30	18	6
30	0	30	19	8	38	27	16	5	35	24	13	2	32	21	10	40	29	18	7	37	26
31	0	31	21	11	1	32	22	12	2	33	23	13	3	34	24	14	4	35	25	15	5
32	0	32	23	14	5	37	28	19	10	1	33	24	15	6	38	29	20	11	2	34	25
33	0	33	25	17	9	1	34	26	18	10	2	35	27	19	11	3	36	28	20	12	4
34	0	34	27	20	13	6	40	33	26	19	12	5	39	32	25	18	11	4	38	31	24
35	0	35	29	23	17	11	5	40	34	28	22	16	10	4	39	33	27	21	15	9	3
36	0	36	31	26	21	16	11	6	1	37	32	27	22	17	12	7	2	38	33	28	23
37	0	37	33	29	25	21	17	13	9	5	1	38	34	30	26	22	18	14	10	6	2
38	0	38	35	32	29	26	23	20	17	14	11	8	5	2	40	37	34	31	28	25	22
39	0	39	37	35	33	31	29	27	25	23	21	19	17	15	13	11	9	7	5	3	1
40	0	40	39	38	37	36	35	34	33	32	31	30	29	28	27	26	25	24	23	22	21

Multiplication table modulo 41, continued:

	21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
2	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
3	22	25	28	31	34	37	40	2	5	8	11	14	17	20	23	26	29	32	35	38
4	2	6	10	14	18	22	26	30	34	38	1	5	9	13	17	21	25	29	33	37
5	23	28	33	38	2	7	12	17	22	27	32	37	1	6	11	16	21	26	31	36
6	3	9	15	21	27	33	39	4	10	16	22	28	34	40	5	11	17	23	29	35
7	24	31	38	4	11	18	25	32	39	5	12	19	26	33	40	6	13	20	27	34
8	4	12	20	28	36	3	11	19	27	35	2	10	18	26	34	1	9	17	25	33
9	25	34	2	11	20	29	38	6	15	24	33	1	10	19	28	37	5	14	23	32
10	5	15	25	35	4	14	24	34	3	13	23	33	2	12	22	32	1	11	21	31
11	26	37	7	18	29	40	10	21	32	2	13	24	35	5	16	27	38	8	19	30
12	6	18	30	1	13	25	37	8	20	32	3	15	27	39	10	22	34	5	17	29
13	27	40	12	25	38	10	23	36	8	21	34	6	19	32	4	17	30	2	15	28
14	7	21	35	8	22	36	9	23	37	10	24	38	11	25	39	12	26	40	13	27
15	28	2	17	32	6	21	36	10	25	40	14	29	3	18	33	7	22	37	11	26
16	8	24	40	15	31	6	22	38	13	29	4	20	36	11	27	2	18	34	9	25
17	29	5	22	39	15	32	8	25	1	18	35	11	28	4	21	38	14	31	7	24
18	9	27	4	22	40	17	35	12	30	7	25	2	20	38	15	33	10	28	5	23
19	30	8	27	5	24	2	21	40	18	37	15	34	12	31	9	28	6	25	3	22
20	10	30	9	29	8	28	7	27	6	26	5	25	4	24	3	23	2	22	1	21
21	31	11	32	12	33	13	34	14	35	15	36	16	37	17	38	18	39	19	40	20
22	11	33	14	36	17	39	20	1	23	4	26	7	29	10	32	13	35	16	38	19
23	32	14	37	19	1	24	6	29	11	34	16	39	21	3	26	8	31	13	36	18
24	12	36	19	2	26	9	33	16	40	23	6	30	13	37	20	3	27	10	34	17
25	33	17	1	26	10	35	19	3	28	12	37	21	5	30	14	39	23	7	32	16
26	13	39	24	9	35	20	5	31	16	1	27	12	38	23	8	34	19	4	30	15
27	34	20	6	33	19	5	32	18	4	31	17	3	30	16	2	29	15	1	28	14
28	14	1	29	16	3	31	18	5	33	20	7	35	22	9	37	24	11	39	26	13
29	35	23	11	40	28	16	4	33	21	9	38	26	14	2	31	19	7	36	24	12
30	15	4	34	23	12	1	31	20	9	39	28	17	6	36	25	14	3	33	22	11
31	36	26	16	6	37	27	17	7	38	28	18	8	39	29	19	9	40	30	20	10
32	16	7	39	30	21	12	3	35	26	17	8	40	31	22	13	4	36	27	18	9
33	37	29	21	13	5	38	30	22	14	6	39	31	23	15	7	40	32	24	16	8
34	17	10	3	37	30	23	16	9	2	36	29	22	15	8	1	35	28	21	14	7
35	38	32	26	20	14	8	2	37	31	25	19	13	7	1	36	30	24	18	12	6
36	18	13	8	3	39	34	29	24	19	14	9	4	40	35	30	25	20	15	10	5
37	39	35	31	27	23	19	15	11	7	3	40	36	32	28	24	20	16	12	8	4
38	19	16	13	10	7	4	1	39	36	33	30	27	24	21	18	15	12	9	6	3
39	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2
40	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1