


***Amherst College***
***Department of Mathematics and Statistics***

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MATH 222 MIDTERM EXAM 29 OCTOBER 2026

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NAME: Solutions

**Read This First!**

- The exam uses **both sides of the page**.
- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back. No other books, notes, calculators, computers, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Show **ALL** work clearly in the space provided or on the blank pages.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- You may use any tactic, and cite any library theorem, that we have used in class or on the problem sets, except where the theorem is essentially the statement you are asked to prove. If you do not remember the exact name of a theorem, state clearly what it says.
- Write Lean code as you would type it. Small slips in syntax will cost you little or nothing, as long as your meaning is clear. All Lean code assumes `import Mathlib`.

**Grading - For Instructor Use Only**

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\sum$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 9 | 6 | 6 | 9 | 6 | 9 | 9 | 9 | 9 | 72 |
| Score: | | | | | | | | | | |

1. Write each of the following statements in Lean. You do not need to prove them.

   (a) [3 points] The only natural numbers dividing 31 are 1 and 31.

```
∀ d : ℕ, d ∣ 31 → d = 1 ∨ d = 31
```

   (b) [3 points] Not every integer is equal to a difference of two perfect squares.

```
¬ ∀ (n : ℤ), ∃ (a b : ℤ), n = a^2 - b^2
```

   (c) [3 points] For every positive real number $\varepsilon$, there exists a positive real number $x$ such that $\frac{1}{x} < \varepsilon$.

```
∀ (ε : ℝ), ε > 0 → ∃ (x : ℝ), x > 0 ∧ 1/x < ε
```

**(next problem on reverse)**

2. [6 points] Complete the following proof, by filling in the steps of the `calc` block. Do not use `linarith` in this problem; deduce the result from simpler techniques. You may use `linarith` on all later problems of the exam, however.

```
theorem linear_system (x y : ℚ) (h1 : y + 2 ≥ 7) (h2 : x - 3*y ≥ 5) :
    x ≥ 20 := by
  calc
    -- Complete the proof from here
```

```
x = (x-3*y) +3*(y+2) -6  := by ring
_ ≥    5  + 3*7  - 6     := by rel [h1, h2]
_ =    20                := by norm_num
```

3. Consider the following incomplete proof.

```
example (y : ℝ) (hy : ∀ x : ℝ, x^2 - 2*x ≥ y) : y ≤ -1 := by
  specialize hy 1
  -- PART A: tactic state here?
  -- PART B: complete the proof from here
```

(a) [3 points] What is the tactic state at the line marked `PART A`?

```
y : ℝ
hy : 1^2 - 2*1 ≥ y
⊢ y ≤ -1
```

(b) [3 points] Complete the proof from that point (it is possible to do this in one line).

```
linarith
```

OR

```
calc
  y ≤ 1^2 - 2*1  := by exact hy
  _ = -1         := by norm_num
```

4. Suppose that the tactic state is currently

```
a b : ℤ
ha : Odd a
hb : Even b
⊢ Odd (3 * a + b)
```

and you then type the following two lines.

```
obtain ⟨k, hk⟩ := ha
obtain ⟨l, hl⟩ := hb
```

(a) [3 points] What is the tactic state now?

a b : ℤ
~~ha:~~
k : ℤ
hk : a = 2*k+1
l : ℤ
hl : b = l+l
⊢ Odd (3*a + b)

(b) [6 points] Complete the proof.

use 3*k+l+1
~~ring~~
rw [hk, hl]
ring -- linarith also works.

scratch
3a+b = 3(2k+1) + 2l
= 6k+2l+2+1
= 2(3k+l+1)+1

**(next problem on reverse)**

5. Consider the following incomplete proof.

```
example (n : ℤ) : ¬ 2 ∣ n → ¬ 4 ∣ n := by
  intro h2
  contrapose! h2 with h4
  -- PART A: tactic state here?
  -- PART B: complete the proof from here
```

(a) [3 points] What is the tactic state at the line marked `PART A`?

$n : \mathbb{Z}$

$h4 : 4 \mid n$

$\vdash \; 2 \mid n$

(b) [3 points] Complete the proof.

```
obtain ⟨k, hk⟩ := h4
use 2 * k
rw [hk]
ring
```

} can also use linarith in place of these two lines

6. [9 points] Prove the following theorem. (This fact may well be in Mathlib under some name, but do not try to guess it: prove it using the techniques we have used in class.)

   *Hint:* in the forward direction, it may help to prove the equation $(x - y) * (x + y) = 0$ as a hypothesis.

```
theorem equal_squares {x y : ℝ} : x^2 = y^2 ↔ x = y ∨ x = -y := by
```

```
constructor
· intro h   -- h: x^2=y^2
  have h1: (x-y)*(x+y) =0 := by
    | calc
    |   (x-y)*(x+y)  =x^2-y^2 := by ring
    |   _            = y^2-y^2 := by rw [h]
    |   _            = 0       := by ring
  obtain diff0 | sum0 := mul_eq_zero.mp h1
  · | -- diff0: x-y =0
    | left
    | linarith
  · | -- sum0: x+y=0
    | right
    | linarith
· intro h   -- h: x=y ∨ x=-y
  obtain eq | eqminus := h
  · | -- eq: x=y
    | rw [eq]
    | ring -- or rfl, or others will work
  · | -- eqminus: x=-y
    | rw [eqminus]
    | ring
```

**(next problem on reverse)**

7. [9 points] Prove the following lemma. (Suggestion: use induction.)

```
lemma pow_ineq (n : ℕ) : 3^n ≥ 1 + 2*n := by
```

```
induction' n with n hn
• |-- ⊢ 3^0 ≥ 1+2·0
  |norm_num
-- ind. step. hn: 3^n ≥ 1+2*n, ⊢ 3^(n+1) ≥ 1+2(n+1)
calc
  |3^(n+1) = 3 * 3^n      := by ring
  |_       ≥ 3·(1+2*n)    := by rel [hn]
  |_       = 1 + 2·(n+1)+4*n := by ring
  |_       ≥ 1+2·(n+1)    := by linarith
```

Alternative

You can also get a 2-line proof in the inductive step:

```
ring      -- turns 3^(n+1) into 3*3^n
linarith [hn]  -- chains 3^n ≥ 1+2n into the goal
```

8. Consider the sequence defined by $a_0 = 2$, $a_1 = 4$, and

$$a_{n+2} = 4a_{n+1} - 3a_n \quad \text{for } n \geq 0.$$

(a) [3 points] Define this sequence in Lean, as a function `a : ℕ → ℤ`.

```
def a (n : ℕ) : ℤ := match n with
| 0 => 2
| 1 => 4
| n+2 => 4*(a (n+1)) - 3*(a n)
```

(b) [6 points] Prove the following lemma in Lean, by filling in the three cases below.

```
lemma a_formula (n : ℕ) : a n = 3^n + 1 := by
  match n with
  | 0 =>
      rw [a]
      norm_num

  | 1 =>
      rw [a]
      norm_num

  | n + 2 =>
      rw [a]
      rw [a_formula n, a_formula (n+1)]
      -- Goal now is ⊢ 4*(3^(n+1)+1) - 3*(3^n + 1) = 3^(n+2)+1
      ring
```

**(next problem on reverse)**

9. [9 points] Suppose you have already proved the following lemma.

```
theorem pos_case (m n : ℤ) (hn : n > 0) : m^2 ≠ 5 * n^2 := by
  -- Proof omitted. You may assume it is done correctly.
```

(Notice that, unlike in a similar question on Quiz 2, the lemma `pos_case` makes no assumption about the sign of `m`.)

Using it, prove the following theorem.

```
theorem no_soln (m n : ℤ) (hn : n ≠ 0) : m^2 ≠ 5 * n^2 := by
```

```
by_cases hn : n > 0
· | exact pos_case m n hn
· | have hn' : (-n) > 0 := by
  |     | ~~linarith~~ omega
  | ~~exact pos_case m (-n)~~
  | have helper : m^2 ≠ 5*(-n)^2 := by
  |     | exact pos_case m (-n) hn'
  | contrapose! helper with h
  | -- now h : m^2 = 5 * n^2, ⊢ m^2 = 5 * (-n)^2
  | rw [h]
  | ring
```

another soln can use wlog instead of by_cases, eg:

```
wlog hn' : n > 0
· | have hn' : (-n) > 0 := by linarith
  | have helper := ~~pos_case~~ this m (-n) hn'   ← (by linarith) supplies hypothesis (-n) ≠ 0
  | contrapose! helper with h
  | rw [h]
  | ring
exact pos_case m n hn'
```