1. Write each of the following statements in Lean. You do not need to prove them.
   (a) [3 points] The only natural numbers dividing 31 are 1 and 31.
   (b) [3 points] Not every integer is equal to a difference of two perfect squares.
   (c) [3 points] For every positive real number ε, there exists a positive real number x such that $\frac{1}{x} < \varepsilon$.

2. [6 points] Complete the following proof, by filling in the steps of the `calc` block. Do not use `linarith` in this problem; deduce the result from simpler techniques. You may use `linarith` on all later problems of the exam, however.

```
theorem linear_system (x y : ℚ) (h1 : y + 2 ≥ 7) (h2 : x - 3*y ≥ 5) :
    x ≥ 20 := by
  calc
    -- Complete the proof from here
```

3. Consider the following incomplete proof.

```
example (y : ℝ) (hy : ∀ x : ℝ, x^2 - 2*x ≥ y) : y ≤ -1 := by
  specialize hy 1
  -- PART A: tactic state here?
  -- PART B: complete the proof from here
```

   (a) [3 points] What is the tactic state at the line marked `PART A`?
   (b) [3 points] Complete the proof from that point (it is possible to do this in one line).

4. Suppose that the tactic state is currently

```
a b : ℤ
 ha : Odd a
 hb : Even b
 ⊢ Odd (3 * a + b)
```

   and you then type the following two lines.

```
  obtain ⟨k, hk⟩ := ha
  obtain ⟨l, hl⟩ := hb
```

   (a) [3 points] What is the tactic state now?
   (b) [6 points] Complete the proof.

5. Consider the following incomplete proof.

```
example (n : ℤ) : ¬ 2 ∣ n → ¬ 4 ∣ n := by
  intro h2
  contrapose! h2 with h4
  -- PART A: tactic state here?
  -- PART B: complete the proof from here
```

   (a) [3 points] What is the tactic state at the line marked `PART A`?
   (b) [3 points] Complete the proof.

6. [9 points] Prove the following theorem. (This fact may well be in Mathlib under some name, but do not try to guess it: prove it using the techniques we have used in class.)

   Hint: in the forward direction, it may help to prove the equation $(x - y) * (x + y) = 0$ as a hypothesis.

   ```
   theorem equal_squares {x y : ℝ} : x^2 = y^2 ↔ x = y ∨ x = -y := by
   ```

7. [9 points] Prove the following lemma. (Suggestion: use induction.)

   ```
   lemma pow_ineq (n : ℕ) : 3^n ≥ 1 + 2*n := by
   ```

8. Consider the sequence defined by $a_0 = 2$, $a_1 = 4$, and

   $$a_{n+2} = 4a_{n+1} - 3a_n \quad \text{for } n \geq 0.$$

   (a) [3 points] Define this sequence in Lean, as a function `a : ℕ → ℤ`.

   (b) [6 points] Prove the following lemma in Lean, by filling in the three cases below.

   ```
   lemma a_formula (n : ℕ) : a n = 3^n + 1 := by
     match n with
     | 0 =>




     | 1 =>




     | n + 2 =>
   ```

9. [9 points] Suppose you have already proved the following lemma.

   ```
   theorem pos_case (m n : ℤ) (hn : n > 0) : m^2 ≠ 5 * n^2 := by
     -- Proof omitted. You may assume it is done correctly.
   ```

   (Notice that, unlike in a similar question on Quiz 2, the lemma `pos_case` makes no assumption about the sign of `m`.)

   Using it, prove the following theorem.

   ```
   theorem no_soln (m n : ℤ) (hn : n ≠ 0) : m^2 ≠ 5 * n^2 := by
   ```