Work on the problems below with one or two students nearby.

Call me over if you have questions or want to check answers!

Each of these problems can be solved using the **pigeonhole principle**:
*Let $m, n \in \mathbb{N}$. Suppose that a set of objects ("pigeons") are distributed among $n$ categories ("pigeonholes"). If the number of objects is at least $n(m-1)+1$, then there is some category with at least $m$ objects in it.*

1. Assume that all people have at most 200,000 hairs on their head. Prove that in New York City, there are two people with exactly the same number of hairs on their head.

There are at least 8 million people in NYC. (current pop. ≈ 8.6 million).

Sort them into pigeonholes $\{0, 1, \cdots, 200{,}000\}$ according ~~the~~ to the number of hairs on their head. Since 8 million ≥ 200,000 + 1, there's a pigeonhole with ≥ 2 people (in fact, there's one with ≥ 40 people). so some two people have the same number of hairs.

**The pigeons are:** The ≥ 8 million people in NYC.

**The pigeonholes are:** The numbers $0, 1, \cdots, 200{,}000$.

2. How many digits of $\pi$ must you examine before you are guaranteed to see some specific digit at least 1000 times?

In the "worst case," you might see each digit $0, 1, \cdots, 9$ 999 times before seeing any 1000 times. But for certain you'll see a repeat once you see

$$10 \cdot 999 + 1 = 9991$$

digits.

**The pigeons are:** Place values in the decimal of $\pi$.

**The pigeonholes are:** The digits $0, 1, \cdots, 9$.

3. Suppose that there are seven bicycles on the section of the Norwottuck Rail Trail running from Amherst College to the Connecticut River. Assume that this section of the trail is exactly six miles long. Prove that there are two bicycles that are at most one mile apart.

By dividing the trail into six one-mile segments, we may divide the seven bicyclists into six categories. Note that we could place the mile marker "1" into either the $1^{st}$ segment or the second, but it won't make a difference.
By the pigeonhole principle, some two bicycles are in the same segment. These two bicycles are at most one mile apart, since the segments are one mile long.

**The pigeons are:**
The seven bicycles.

**The pigeonholes are:**
The sets $[0,1]$, $(1,2]$, $(2,3]$, $(3,4]$, $(4,5]$, $(5,6]$.
(or $[0,1)$, $[1,2)$, etc. if endpoints are treated differently).

4. Suppose that $S = \{s_1, s_2, \cdots, s_{10}\}$ is a set of 10 natural numbers. Prove that some nonempty subset of $S$ has sum divisible by 10. By convention, the "sum" of a set with just one number is the number itself.

This is a more challenging application of pigeonhole, so I've told you below what the pigeonholes and pigeonholes could be below; see if you can find a proof along these lines.

Consider the <u>eleven</u> numbers $0$, $s_1$, $s_1+s_2$, $\cdots$, $s_1+s_2+\cdots+s_{10}$.
Sort them by their last digit. There are <u>ten</u> (< eleven) digits possible, so some two sums end with the same digit. Their difference is

$$\cancel{\text{Start}}\; s_{m+1} + s_{m+2} + \cdots + s_n$$

for some $m, n \in \{0, \cdots 10\}$ w/ $m < n$, and must end in a 0, hence ~~the sum~~ this is a sum of a subset $\{s_{m+1}, \cdots, s_n\} \subseteq S$ that is divisible by 10.

"more detail on this part will be possible after we discuss the division algorithm"

**The pigeons are:** The numbers $0$, $s_1$, $s_1 + s_2$, $s_1 + s_2 + s_3, \cdots, s_1 + s_2 + \cdots + s_{10}$.

**The pigeonholes are:** The digits $0, 1, 2, \cdots, 9$.