PSet 9 Solutions

① Use the center $x=100$ (since $\sqrt{100}=10$).

$$f(x) = \sqrt{x} \qquad\qquad f(100) = 10$$

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad\qquad f'(100) = \frac{1}{2\cdot 10} = \frac{1}{20}$$

$$f''(x) = -\frac{1}{4x^{3/2}} \qquad\qquad f''(100) = -\frac{1}{4\cdot 1000} = -\frac{1}{4000}$$

So the quad. approx. is

$$P_2(x) = 10 + \frac{1}{20}(x-1) - \frac{1}{2}\cdot\frac{1}{4000}\cdot(x-1)^2$$

$$= 10 + \frac{x-1}{20} - \frac{x-1}{8000}$$

This gives the following approximation for $\sqrt{101}$:

$$P_2(101) = 10 + \frac{1}{20} - \frac{1}{8000}$$

$$= 10 + 0.05 - 0.000125$$

$$= \underline{10.049875}.$$

The exact value of $\sqrt{101}$ is $10.0498756211\ldots$, so the error of this approximation is about 6×10^{-7} (~~the~~ less than one millionth).

(2)

a) $\sum_{n=0}^{\infty} \frac{2^n}{3^n \cdot n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{2}{3}\right)^n = \boxed{e^{2/3}}.$

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(\frac{4}{3}-1\right)^n = \ln\left(\frac{4}{3}\right)$

$= \boxed{\ln 4 - \ln 3}.$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,2^{2n}}$

$= 2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \left(\frac{1}{2}\right)^{2n+1}$

$= \boxed{2 \cdot \sin\left(\frac{1}{2}\right)}$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \cdot 4^n} = 2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \cdot 2^{2n+1}}$

$= 2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cdot \left(\frac{1}{2}\right)^{2n+1}$

$= \boxed{2 \tan^{-1}\left(\frac{1}{2}\right)}$

(3) a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\sqrt{x})^{2n}$

$= \frac{1}{\sqrt{x}} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (\sqrt{x})^{2n+1}$

$= \boxed{\frac{1}{\sqrt{x}} \cdot \sin(\sqrt{x})}$

(not part of the course)

[Note: If $x<0$, this argument is not valid since $\sqrt{x}$ isn't real. However, you can compute the sum in this case as follows:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(-(\sqrt{-x})^2\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(-1)^n \cdot (\sqrt{-x})^{2n}$$

$$= \frac{1}{\sqrt{-x}}\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(\sqrt{-x})^{2n+1}$$

$$= \frac{1}{\sqrt{-x}} \cdot \frac{1}{2} \cdot \left(e^{\sqrt{-x}} - e^{-\sqrt{-x}}\right)$$

where you observe in the last step that $e^{x} - e^{-x} = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$.

I would not expect you to make this observation on a homework or exam problem.]

b) $$\sum_{n=0}^{\infty} \frac{1}{n!} x^{3n} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^3)^n = \boxed{e^{x^3}}$$

c) $$\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n)!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (3x)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{3x})^{2n}$$

$$= \boxed{\cos(\sqrt{3x})}$$

[Note. Like in part (a), you must assume $x \geq 0$. Otherwise x^n would be $(-(\sqrt{-x})^2)^n = (-1)^n \cdot (\sqrt{-x})^{2n}$ and the sum would be $\frac{1}{2}\left(e^{\sqrt{-3x}} + e^{-\sqrt{-3x}}\right)$.]

d) $$\sum_{n=1}^{\infty} \frac{e^7}{n!}(x-7)^n = e^7 \cdot \left[\sum_{n=1}^{\infty} \frac{1}{n!}(x-7)^n\right]$$

$$= e^7 \cdot \left[\sum_{n=0}^{\infty} \frac{1}{n!}(x-7)^n - \frac{1}{0!}\cdot(x-7)^0\right]$$

$$= e^7\left(e^{x-7}-1\right)$$

$$= \boxed{e^x - e^7}$$

(4)

a) $$\frac{1}{1-x} = 1+x+x^2+\cdots$$

$$= \boxed{\sum_{n=0}^{\infty} x^n}$$ (geometric series)

b) $$x^2e^x = x^2\cdot\sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{1}{n!}\cdot x^{n+2}} \quad \left(\text{or } \sum_{n=2}^{\infty}\frac{1}{(n-2)!}x^n\right)$$

c) The derivs. of $\sin x$ follow the pattern

$\sin x,\ \cos x,\ -\sin x,\ -\cos x,\ \sin x,\ \cos x,\ -\sin x,\ -\cos x,\ \ldots$

~~Then~~ Hence when evaluated at $\pi/2$ these follow the pattern:

$1, 0, -1, 0, 1, 0, -1, 0, \ldots$

so the Taylor series @ $x=\pi/2$ is

$$1 \cancel{+} - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\cdot\left(x-\frac{\pi}{2}\right)^2 - \frac{1}{6!}\left(x-\frac{\pi}{2}\right)^6 + \cdots$$

$$\boxed{= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}}$$

Alternatively, just notice that

$$\sin x = \cos\left(\tfrac{\pi}{2}-x\right) = \cos\left(x-\tfrac{\pi}{2}\right)$$

and then substitute $x-\frac{\pi}{2}$ for x in the Taylor series of $\cos x$ around $x=0$.

d)

$$\int_0^x e^{-t^2}\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{1}{n!}(-t^2)^n\,dt$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}\int_0^x t^{2n}\,dt\right]$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\cdot(2n+1)}\cdot x^{2n+1}}$$

⑤ $$e^{2x} = 1+(2x)+\frac{1}{2}(2x)^2+\frac{1}{6}(2x)^3+\frac{1}{24}(2x)^4+\frac{1}{120}(2x)^5+\frac{1}{720}(2x)^6+\dots$$

$$= 1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4+\frac{4}{15}x^5+\frac{4}{45}x^6+\dots$$

and

$$\tan^{-1}x = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\dots$$

Hence

$$e^{2x}\tan^{-1}x = \left(1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4+\frac{4}{15}x^5+\frac{4}{45}x^6+\dots\right)\cdot\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\dots\right)$$

$$= [1\cdot x]+[2x\cdot x]+\left[2x^2\cdot x+1\cdot\left(-\tfrac{1}{3}x^3\right)\right]$$
$$+\left[\tfrac{4}{3}x^3\cdot x+2x\cdot\left(-\tfrac{1}{3}x^3\right)\right]+\left[\tfrac{2}{3}x^4\cdot x+2x^2\cdot\left(-\tfrac{1}{3}x^3\right)+1\cdot\left(\tfrac{1}{5}x^5\right)\right]$$
$$+\left[\tfrac{4}{15}x^5\cdot x+\tfrac{4}{3}x^3\cdot\left(-\tfrac{1}{3}x^3\right)+2x\cdot\left(\tfrac{1}{5}x^5\right)\right]+\dots$$

$$= x + 2x^2 + \left[2-\tfrac{1}{3}\right]x^3 + \left[\tfrac{4}{3} - \tfrac{2}{3}\right]x^4 + \left[\tfrac{2}{3} - \tfrac{2}{3} + \tfrac{1}{5}\right]x^5 + \left[\tfrac{4}{15} - \tfrac{4}{9} + \tfrac{2}{5}\right]x^6 + \cdots$$

$$= x + 2x^2 + \tfrac{5}{3}x^3 + \tfrac{2}{3}x^4 + \tfrac{1}{5}x^5 + \tfrac{2}{9}x^6 + \cdots$$

so the 6th order Taylor approximation consists of the terms up to the x^6 term of the Taylor series, i.e.

$$\boxed{P_6(x) = x + 2x^2 + \tfrac{5}{3}x^3 + \tfrac{2}{3}x^4 + \tfrac{1}{5}x^5 + \tfrac{2}{9}x^6}$$

(6)

$$\int_0^1 e^{-x^2/2}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\tfrac{1}{2}x^2\right)^n dx$$

$$= \sum_{n=0}^{\infty}\left[\int_0^1 \frac{(-1)^n}{n!\cdot 2^n}\, x^{2n}\,dx\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!\cdot 2^n}\cdot\frac{1}{2n+1}\cdot x^{2n+1}\right]_0^1$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\cdot 2^n\cdot(2n+1)}}$$

(7) a) Ratio test: $L = \lim_{n\to\infty}\left|\frac{(n+1)^3 x^{2n+2}/(n+1)!}{n^3\cdot x^{2n}/n!}\right| = \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^3\cdot\frac{n!}{(n+1)!}\cdot\frac{x^{2n+2}}{x^{2n}}\right|$

$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^3\cdot\frac{x^2}{n+1}\right| = |x^2|\cdot\lim_{n\to\infty}\frac{(n+1)^2}{n^3} = 0.$$

So the series converges for all x, ie. the radius of conv. is $\boxed{\infty}$.

b) Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}/100^{n+1}}{n! \cdot x^n / 100^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{x^{n+1}}{x} \cdot \frac{100^n}{100^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (n+1) \cdot \frac{x}{100} \right| = \infty. \quad (\text{for } x \neq 0)$$

So the series only converges for $x=0$; the rad. of conv. is $\boxed{0}$.

c) Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{7^{n+1} \cdot x^{n+1} / (8n+22)}{7^n \cdot x^n / (8n+14)} \right|$$

$$= \lim_{n\to\infty} \left| 7 \cdot x \cdot \frac{8n+14}{8n+22} \right|$$

$$= 7 \cdot |x| \cdot \lim_{n\to\infty} \left(\frac{8n+14}{8n+22} \right)$$

$$= 7 \cdot |x|$$

So the radius of convergence is $\boxed{1/7}$.

d) Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1} / (n+1)^{n+1}}{n!\, x^n / n^n} \right|$$

$$= \lim_{n\to\infty} \left| (n+1) \cdot x \cdot \frac{n^n}{(n+1)^{n+1}} \right|$$

$$= |x| \cdot \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n$$

Now observe that

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = e^{\lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right)}$$

and

$$\lim_{n\to\infty}\left[n\cdot \ln\left(\frac{n}{n+1}\right)\right] = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{1/n}$$

$$= \lim_{n\to\infty} \frac{(1\cdot(n+1)-n\cdot 1)/(n+1)^2 \cdot \frac{n+1}{n}}{-1/n^2} \quad \text{(l'hôpital)}$$

$$= \lim_{n\to\infty}\left(-\frac{n^2}{(n+1)^2}\right) = -1$$

hence $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = e^{-1}$ and $L = |x|\cdot e^{-1}$.

So the radius of convergence is $\boxed{e}$.

8 Differentiate:

$$f'(x) = \sum_{n=0}^{\infty} a_n \cdot n x^{n-1}$$

and mult. by x:

$$x\cdot f'(x) = \sum_{n=0}^{\infty} a_n\cdot n\cdot x^n$$

Hence $\sum_{n=0}^{\infty} n\cdot a_n x^n$ is the Taylor series @ $x=0$

of $\boxed{x\cdot f'(x)}$.

9

a)

$$Q''(0) = \cos 0 - 2Q'(0) - Q(0)$$
$$= 1 - 2\cdot 0 - 0 = \boxed{1}.$$ (using the initial conditions)

b)

$$Q''' + 2\cdot Q'' + Q' = -\sin t$$
$$\Rightarrow \quad Q'''(0) = -\sin 0 - 2Q''(0) - Q'(0)$$
$$= 0 - 2\cdot 1 - 0$$
$$= \boxed{-2}$$

c)

$$Q^{(4)} + 2\cdot Q''' + Q'' = -\cos t$$
$$\Rightarrow \quad Q^{(4)}(0) = -\cos 0 - 2\cdot Q'''(0) - Q''(0)$$
$$= -1 - 2\cdot(-2) - 1$$
$$\boxed{Q^{(4)}(0) = 2}$$

and ~~sin t~~

$$Q^{(5)} + 2\cdot Q^{(4)} + Q''' = \sin t$$
$$\Rightarrow \quad Q^{(5)}(0) = \sin 0 - 2\cdot Q^{(4)}(0) - Q'''(0)$$
$$= 0 - 2\cdot 2 - (-2)$$
$$\boxed{Q^{(5)}(0) = -2}$$

d)

$$P_5(t) = Q(0) + Q'(0)t + \frac{Q''(0)}{2}t^2 + \frac{Q'''(0)}{6}t^3 + \frac{Q^{(4)}(0)}{24}t^4 + \frac{Q^{(5)}(0)}{120}t^5$$
$$= 0 + 0\cdot t + \frac{1}{2}t^2 + -\frac{2}{6}t^3 + \frac{2}{24}t^4 - \frac{2}{120}t^5$$
$$= \boxed{\frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{12}t^4 - \frac{1}{60}t^5}$$

e) Using this approximation:

$$Q(1) \approx P_5(1) = \frac{1}{2} - \frac{1}{3} + \frac{1}{12} - \frac{1}{60}$$

$$= \boxed{7/30}$$

$$= 0.23333\ldots$$

(the exact value of $Q(1)$ is about 0.2368).