Goal Practice verifying convergence / divergence with geometric series test (GST) and nth term divergence test (NTDT). Evaluate some sums of series using the geometric series formula.
Reference: §11.2.

Examples to study first

In each example: **determine whether the given Series Converges or Diverges. If it Converges, find the Sum value. Justify.**

Example $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 5^{n-1}}{3^{2n+1}} = -\frac{1}{3^3} + \frac{5}{3^5} - \frac{5^2}{3^7} + \frac{5^3}{3^9} + \dots$

Solution This is geometric, with frist term $a = -\dfrac{1}{27}$ and common ratio $r = -\dfrac{5}{3^2} = -\dfrac{5}{9}$.

The series **Converges by Geometric Series Test (GST)**, because $|r| = \left|-\dfrac{5}{9}\right| = \dfrac{5}{9} < 1$.

The sum is $\displaystyle\frac{a}{1-r} = \frac{-\frac{1}{27}}{1-\left(-\frac{5}{9}\right)} = \frac{-\frac{1}{27}}{\frac{14}{9}} = -\frac{1}{{}_3\cancel{27}} \cdot \frac{\cancel{9}}{14} = -\frac{1}{3}\cdot\frac{1}{14} = \boxed{-\frac{1}{42}}$

Example $\displaystyle\sum_{n=0}^{\infty} \left(\frac{7}{3}\right)^n = 1 + \frac{7}{3} + \frac{7^2}{3^2} + \frac{7^3}{3^3} + \dots$

Solution Here $a = 1$ and $r = \dfrac{7}{3}$.

The series **Diverges by GST**, because $|r| = \dfrac{7}{3} \geq 1$.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

Solution The series **Diverges by the n^{th} Term Divergence Test (nTDT)** because

$$\lim_{n\to\infty} \frac{e^n}{n^2}\,\overset{\frac{\infty}{\infty}}{} = \lim_{x\to\infty} \frac{e^x}{x^2}\,\overset{\frac{\infty}{\infty}}{}\ \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{2x}\,\overset{\frac{\infty}{\infty}}{}\ \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\cancel{e^x}^{\,\infty}}{2} = \infty \neq 0$$

Example $\displaystyle\sum_{n=1}^{\infty} 3$

Solution The series **diverges by nTDT** because $\displaystyle\lim_{n\to\infty} 3 = 3 \neq 0$.

Example $\displaystyle\sum_{n=1}^{\infty} e^{\frac{1}{n}}$

Solution The series **Diverges by nTDT** because $\displaystyle\lim_{n\to\infty} \overset{\nearrow 1}{e^{\overset{\nearrow 0}{\frac{1}{n}}}} = 1 \neq 0$

Problems to hand in

Determine whether each of the following Converge or Diverge. Justify.

1. $\{8\}_{n=1}^{\infty}$

2. $\displaystyle\sum_{n=1}^{\infty} 8$

3. $\displaystyle\left\{\frac{2n}{3n+1}\right\}_{n=1}^{\infty}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n+1}$

Determine whether the given series Converges or Diverges. If it converges, find the Sum value. Justify.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{8}{5^n}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{8}{5^n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{9^{n-1}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+1}}{3^n}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{4^{2n+1}}{3^{3n-1}}$

10. $\displaystyle\sum_{n=1}^{\infty} e^n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(1999)^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1999}$

14. $\displaystyle\sum_{n=1}^{\infty} \arctan n$

15. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{\ln n}$

16. $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{\pi n^4+1}{3n^4+5}\right)$

17. $\displaystyle\sum_{n=1}^{\infty} \left(1+\ln\left(1+\frac{5}{n}\right)\right)^n$

Consider these variable versions of Geometric Series. Find the values of x for which the series Converges. Find the sum of the Series for those values of x (answer in terms of x).

18. $\displaystyle\sum_{n=1}^{\infty} (-5)^n x^n$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$