Goal Practice taking limits of infinite **sequences**. Master the various notations used for sequences. Practice a bit with L'Hôpital's rule.
Reference: §11.1

Examples to study first

In each example, **determine whether the given sequence Converges or Diverges. If it converges, find the Limit.**

Example $\left\{\dfrac{\ln n}{n^3}\right\}_{n=1}^{\infty}$

Solution $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n^3}^{\frac{\infty}{\infty}} = \lim_{x\to\infty}\frac{\ln x}{x^3} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{3x^2} = \lim_{x\to\infty}\frac{1}{3x^3} = \boxed{0}$ Converges

Note In the calculation above, the first step changes the n to an x. This is to emphasize that the limit of the **sequence** is the same as the limit of the **function** with the same formula. This is an important conceptual point, and it is also done in the examples below. I am not picky about whether this is shown in your work, however.

Example $\left\{\dfrac{e^n}{n^2}\right\}_{n=1}^{\infty}$

Solution $\displaystyle\lim_{n\to\infty}\frac{e^n}{n^2}^{\frac{\infty}{\infty}} = \lim_{x\to\infty}\frac{e^x}{x^2} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = \boxed{\infty}$ Diverges

Example $\left\{\dfrac{4-9n^3}{5n^3+8n^2-7n-6}\right\}_{n=1}^{\infty}$

Solution You have a choice in this problem: you can use L'Hôpital's rule, or you can do the algebra shown below (useful for stacks of polynomials). The algebra is quicker!

$$\lim_{n\to\infty}\frac{4-9n^3}{5n^3+8n^2-7n-6}\,\frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3}\right)} = \lim_{n\to\infty}\frac{\overset{0}{\cancel{\frac{4}{n^3}}}-9}{5+\underset{0}{\cancel{\frac{8}{n}}}-\underset{0}{\cancel{\frac{7}{n^2}}}-\underset{0}{\cancel{\frac{6}{n^3}}}} = \boxed{-\frac{9}{5}}\ \text{Converges}$$

Example $\left\{\left(1-\sin\left(\frac{6}{n^3}\right)\right)^{n^3}\right\}_{n=1}^{\infty}$

Solution $\lim_{n\to\infty}\left(1-\sin\left(\frac{6}{n^3}\right)\right)^{n^3} \overset{1^\infty}{=} \lim_{x\to\infty}\left(1-\sin\left(\frac{6}{x^3}\right)\right)^{x^3}$

$$= e^{\lim_{x\to\infty}\ln\left[\left(1-\sin\left(\frac{6}{x^3}\right)\right)^{x^3}\right]} = e^{\lim_{x\to\infty} x^3\ln\left(1-\sin\left(\frac{6}{x^3}\right)\right)} \overset{\infty\cdot 0}{=} e^{\lim_{x\to\infty}\frac{\ln\left(1-\sin\left(\frac{6}{x^3}\right)\right)^{\frac{0}{0}}}{\frac{1}{x^3}}}$$

$$\overset{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{\cancelto{1}{1-\sin\left(\frac{6}{x^3}\right)}}\cdot\left(\cancelto{-1}{-\cos\left(\frac{6}{x^3}\right)}\right)\cdot\left(-\frac{\cancelto{6}{18}}{\cancel{x^4}}\right)}{-\frac{\cancel{3}}{\cancel{x^4}}}} = e^{-6} = \boxed{\frac{1}{e^6}} \text{ Converges}$$

Example $\left\{\frac{(3n-1)!}{(3n+1)!}\right\}_{n=1}^{\infty}$

Solution $\lim_{n\to\infty}\frac{(3n-1)!}{(3n+1)!} = \lim_{n\to\infty}\frac{\cancel{(3n-1)!}}{(3n+1)(3n)\cancel{(3n-1)!}} = \lim_{n\to\infty}\frac{1}{\cancelto{\infty}{(3n+1)(3n)}} \overset{0}{=} \boxed{0}$ Converges

Problems to hand in

List the first five terms of the Sequence. (Start with $n = 1$)

1. $a_n = \dfrac{(-1)^{n-1}}{5^n}$
2. $a_n = \dfrac{1}{(n+1)!}$
3. $a_n = \dfrac{(-1)^n\ n^2}{n+1}$

Determine whether the given sequence Converges or Diverges. If it converges, find the Limit. Justify your answer.

4. $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$
5. $\left\{\dfrac{5n^2+3}{2n^2-7n}\right\}_{n=1}^{\infty}$
6. $\left\{\dfrac{3n^4-n-5}{7n^4+n^2-9}\right\}_{n=1}^{\infty}$
7. $\left\{\dfrac{\tan^{-1} n}{n}\right\}_{n\geq 1}$
8. $\left\{\dfrac{n^2}{e^n}\right\}_{n\geq 1}$
9. $\left\{n \sin\left(\dfrac{1}{n}\right)\right\}_{n\geq 1}$
10. $\left\{\dfrac{(\ln n)^2}{n}\right\}_{n=1}^{\infty}$
11. $\left\{\dfrac{n^{99}}{\ln n}\right\}_{n=2}^{\infty}$
12. $\left\{\dfrac{\ln(99)}{n^{99}}\right\}_{n\geq 1}$
13. $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}_{n=1}^{\infty}$
14. $\left\{\left(1-\dfrac{5}{n^6}\right)^{n^6}\right\}_{n=1}^{\infty}$
15. $\left\{\left(1-\arcsin\left(\dfrac{3}{n^2}\right)\right)^{n^2}\right\}_{n\geq 1}$
16. $\left\{\ln(2n^2+1)-\ln(n^2+1)\right\}_{n\geq 1}$
17. $\left\{\dfrac{(n+3)!}{(n+1)!}\right\}_{n=1}^{\infty}$
18. $\left\{\dfrac{(2n-1)!}{(2n+1)!}\right\}_{n\geq 1}$
19. $\left\{\cos^2\left(\dfrac{\pi n^6+6}{6n^6+1}\right)\right\}_{n=1}^{\infty}$
20. $\left\{\arctan\left(\dfrac{5n^7+1}{5n^7+7}\right)\right\}_{n=1}^{\infty}$