*Amherst College*
*Department of Mathematics and Statistics*

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**MATH 105** | **TEST #2** | **FALL 2015**

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NAME: Solutions

**Read This First!**

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted. Cell phones out of sight.
- Please read each question carefully. Show **ALL** work clearly in the space provided. You may use the backs of pages for additional work space.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable
- Answers must be clearly labeled in the spaces provided after each question.
- The exam consists of Questions 1–8, which total to 100 points.

**Grading - For Instructor Use Only**

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Points: | 15 | 14 | 10 | 15 | 16 | 10 | 15 | 5 | 100 |
| Score: | | | | | | | | | |

1. [15 points] Compute the following derivatives.

(a) $\frac{d}{dx}\left(\frac{x^2+\pi^2}{x^3+\sqrt{7}^3}\right)$. Do *not* simplify your answer.

$$\frac{\left[\frac{d}{dx}(x^2+\pi^2)\right](x^3+\sqrt{7}^3)-(x^2+\pi^2)\left[\frac{d}{dx}(x^3+\sqrt{7}^3)\right]}{(x^3+\sqrt{7}^3)^2}$$

$$=\boxed{\frac{2x(x^3+\sqrt{7}^3)-(x^2+\pi^2)\cdot 3x^2}{(x^3+\sqrt{7}^3)^2}}$$

(b) Let $f(u)=\frac{h(u)}{u^2+1}$, where $h(2)=-1$ and $h'(2)=3$. Compute $f'(2)$.

$$f'(u)=\frac{h'(u)(u^2+1)-h(u)\cdot \overbrace{2u}^{=(u^2+1)'}}{(u^2+1)^2}$$

$$f'(2)=\frac{h'(2)(2^2+1)-h(2)\cdot 2\cdot 2}{(2^2+1)^2}$$

$$=\frac{3\cdot 5-(-1)\cdot 4}{5^2}=\boxed{19/25}$$

(c) $\left((x^2+1)^3(1-3x)^2\right)'$. Do *not* simplify your answer.

$$=\left[3(x^2+1)^2\frac{d}{dx}(x^2+1)\right](1-3x)^2+(x^2+1)^3\cdot\left[2(1-3x)\frac{d}{dx}(1-3x)\right]$$

$$=\boxed{3(x^2+1)^2\cdot 2x\cdot(1-3x)^2+(x^2+1)^3\cdot 2(1-3x)\cdot(-3)}$$

2. [14 points]

(a) State the limit definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Compute $\frac{d}{dx}\left(\frac{1}{x^2+1}\right)$ using the limit definition of derivative.

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2+1} - \frac{1}{x^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(x^2+1) - ((x+h)^2+1)}{[(x+h)^2+1]\cdot[x^2+1]}}{h}$$

$$= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{[(x+h)^2+1]\cdot[x^2+1]\cdot h}$$

$$= \lim_{h\to 0} \frac{x^2 - (x^2+2xh+h^2)}{[\;\cdots\;]} = \lim_{h\to 0} \frac{-2xh - h^2}{[\;\cdots\;]}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{[(x+h)^2+1]\,[x^2+1]\cdot \cancel{h}} = -\frac{2x+0}{[(x+0)^2+1][x^2+1]}$$

$$= \boxed{-\frac{2x}{(x^2+1)^2}}$$

3. [10 points] Compute the second derivative of $f(x) = \frac{x^2}{x+3}$ and simplify your answer.

$$f'(x) = \frac{2x\cdot(x+3) - x^2\cdot 1}{(x+3)^2}$$

$$= \frac{2x^2+6x-x^2}{(x+3)^2} = \frac{x^2+6x}{(x+3)^2}$$

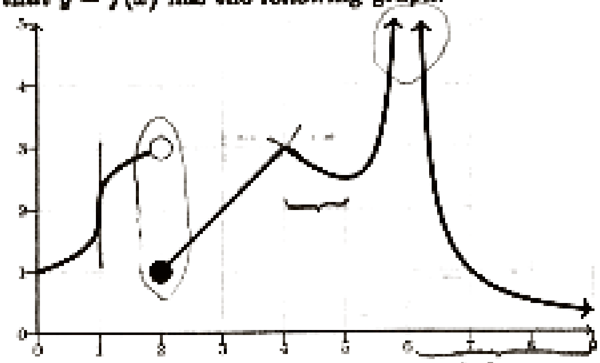
$$f''(x) = \frac{(2x+6)(x+3)^2 - (x^2+6x)\cdot 2(x+3)\cdot 1}{[(x+3)^2]^2}$$

$$= \frac{2\cancel{(x+3)\cdot}[(x+3)^2 - (x^2+6x)]}{(x+3)^{\cancel{4}3}}$$

$$= \frac{2[\cancel{x^2}+\cancel{6x}+9-\cancel{x^2}-\cancel{6x}]}{(x+3)^3}$$

$$= \boxed{\frac{18}{(x+3)^3}}$$

4. [15 points] Suppose that $y = f(x)$ has the following graph:



(a) For which numbers $a$ does $f(x)$ fail to be continuous at $a$? Give reasons using the definition of continuity?

$x=2$ (jump) since $\underset{=3}{\lim_{x\to 2^-} f(x)} \neq \underset{=1}{\lim_{x\to 2^+} f(x)}$.

& $x=6$ (infinite) since $\lim_{x\to 6} f(x) = \infty$.

(b) For which numbers $a$ does $f(x)$ fail to be differentiable at $a$? Give reasons.

$x=2$ & $x=6$ (discontinuities)

& also $x=4$ (a "corner")

& $x=1$ (vertical tangent line)

(c) Find all $x$'s for which $f'(x) > 0$.

all $x$ in $(4,5)$ and $(6,9)$

(where the graph is decreasing).

5. [16 points] We are adding trash to a brand new landfill. Assume that the amount of trash in the landfill at time $t$ (= months since the landfill opened) is given by the formula

$$W(t) = 100t + 10t^2 \text{ tons of trash.}$$

(a) How much trash was added to the landfill during the first six months of its operation?

$W(6) - W(0)$

$= [600 + 360] - 0$

$= \underline{960 \text{ tons}}$

(b) Compute the rate of adding trash during this six month time period.

$\frac{960 \text{ tons}}{6 \text{ months}} = \underline{160} \text{ tons/month}$

(c) What was the rate of adding trash exactly six months after the landfill opened?

$W'(t) = 100 + 20t$

$W'(6) = 100 + 120 = \underline{220} \text{ tons/month}$

(d) When you compare the answers to parts (b) and (c), what conclusion do you draw?

Trash is being added more quickly now than before.

6. [10 points] Find the equation of the line tangent to the curve $y = \dfrac{x^2 + \sqrt{x} + 1}{2 - x}$ at the point where the $x$-coordinate is equal to 1.

$$\frac{dy}{dx} = \frac{\left(2x + \frac{1}{2\sqrt{x}}\right)(2-x) - (x^2 + \sqrt{x} + 1)(-1)}{(2-x)^2}$$

@ $x=1$, this is $\dfrac{(2+\frac{1}{2})\cdot 1 - 3(-1)}{1^2} = \dfrac{11}{2}$.

y-coord. @ $x=1$ is $y = \dfrac{1+1+1}{2-1} = 3$.

So T. line is

$$y - 3 = \frac{11}{2}(x-1)$$

$$= \frac{11}{2}x - \frac{11}{2}$$

$$\Leftrightarrow \boxed{y = \frac{11}{2}x - \frac{5}{2}}$$

7. [15 points] Let

$$f(x) = \frac{x^2}{\sqrt{x^2-1}}.$$

Note that $f(x)$ is defined when $x^2 > 1$, which holds when either $x > 1$ or $x < -1$.

(a) Compute $f'(x)$ and simplify your answer as much as possible. Your final answer should be $f'(x) = \frac{x(x^2-2)}{(x^2-1)^{3/2}}$. To get full credit, I need to see every step of the simplification.

$$f'(x) = \frac{\frac{d}{dx}(x^2)\sqrt{x^2-1} - x^2\frac{d}{dx}\sqrt{x^2-1}}{(\sqrt{x^2-1})^2}$$

$$= \frac{2x\sqrt{x^2-1} - x^2\cdot\frac{1}{2\sqrt{x^2-1}}\cdot 2x}{x^2-1} \quad \frac{d}{dx}(x^2-1)$$

$$= \frac{2x\sqrt{x^2-1}\cdot\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}} - \frac{x^3}{\sqrt{x^2-1}}}{x^2-1} = \frac{\frac{2x(x^2-1)-x^3}{\sqrt{x^2-1}}}{x^2-1}$$

$$= \frac{2x^3-2x-x^3}{\sqrt{x^2-1}\cdot(x^2-1)} = \frac{x^3-2x}{(x^2-1)^{3/2}}$$

$$= \boxed{\frac{x(x^2-2)}{(x^2-1)^{3/2}}}$$

(b) Find all points on the curve where the tangent line is horizontal.

$$f'(x) = 0 \iff x(x^2-2) = 0$$

$$\iff x = 0 \text{ or } \pm\sqrt{2}$$

$f(0) = 0$

$f(\pm\sqrt{2}) = \frac{2}{\sqrt{2-1}} = 2$

$\boxed{(0,0),\ (\sqrt{2},2),\ (-\sqrt{2},2).}$

8. [5 points] The production $q$ of a company depends on both the capital investment $K$ (in dollars) and the size of the labor force $L$ (the number of workers). In economics, one frequently used formula for $q$ in terms of $K$ and $L$ is the *Cobb-Douglas production function*

$$q = \sqrt{KL}.$$

Assuming the capital investment remains constant, compute the rate of change of production as the number of workers increases.

K is constant & L is the variable.

So the desired rate is

$$\frac{dq}{dL} = \frac{d}{dL}\sqrt{KL}$$

$$= \frac{1}{2\sqrt{KL}} \cdot \frac{d}{dL}(KL)$$

$$= \frac{1}{2\sqrt{KL}} \cdot K = \boxed{\frac{\sqrt{K}}{2\sqrt{L}}}$$

$$\left(\text{or } \frac{1}{2}\sqrt{\frac{K}{L}}\right).$$