Amherst College
Department of Mathematics and Statistics

MATH 105 | TEST #1 | FALL 2015

NAME: Solutions

Read This First!

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted. **Cell phones out of sight.**
- Please read each question carefully. Show **ALL** work clearly in the space provided. You may use the backs of pages for additional work space.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable
- Answers must be clearly labeled in the spaces provided after each question.
- The exam consists of Questions 1–6, which total to 100 points. Question 7 is a bonus question (5 points extra credit) that is optional.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	7	Total
Points:	24	18	20	15	15	8	0	100
Score:								

1. [24 points] Compute the following limits. If $+\infty$ or $-\infty$ is a correct answer, please give it.

(a) $\lim_{x\to 1} \frac{1+x^2}{1+x}$

$$= \frac{1+1^2}{1+1} \quad \text{(DSP)}$$

$$= \boxed{1}$$

(b) $\lim_{x\to 1} \frac{x-1}{x^2+x-2}$ Trying DSP: $\frac{1-1}{1^2+1-2} = 0/0.$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}}{(x+2)\cancel{(x-1)}}$$

$$= \lim_{x\to 1} \frac{1}{x+2} \overset{\text{DSP}}{=} \frac{1}{1+2} = \boxed{1/3}$$

(c) $\lim_{x\to 4^+} \frac{x^2-1}{4-x}$

$$= \frac{4^2-1}{4-4} = \frac{17}{0^-}$$ because for $x>4$, $4-x<0$.

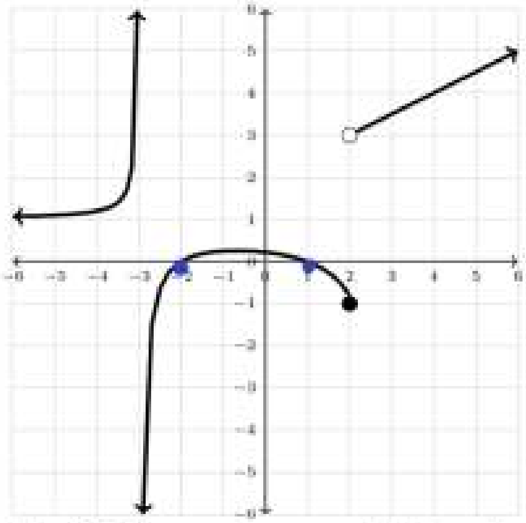
$$= \boxed{-\infty}$$

(d) $\lim_{x\to 1} \frac{\sqrt{x^2+1}-\sqrt{2}}{x-1} \cdot \frac{\sqrt{x^2+1}+\sqrt{2}}{\sqrt{x^2+1}+\sqrt{2}}$

$$= \lim_{x\to 1} \frac{(x^2+1)-2}{(x-1)(\sqrt{x^2+1}+\sqrt{2})} = \lim_{x\to 1} \frac{x^2-1}{(x-1)(\sqrt{x^2+1}+\sqrt{2})}$$

$$= \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{x^2+1}+\sqrt{2})} = \frac{1+1}{\sqrt{1+1}+\sqrt{2}} = \frac{2}{2\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}} \left(\text{or } \frac{\sqrt{2}}{2}\right)$$

2. [18 points] Consider the following graph:



(a) What is the domain of f? Express your answer in interval notation.

All x except $x=-3$, ie. $\boxed{(-\infty,-3)\cup(-3,\infty)}$

(b) For which x's is $f(x)=0$?

$x=-2$ & $x=1$.

(c) For which x's is $f(x)<0$? Express your answer in interval notation.

$(-3,-2)\cup(1,2]$

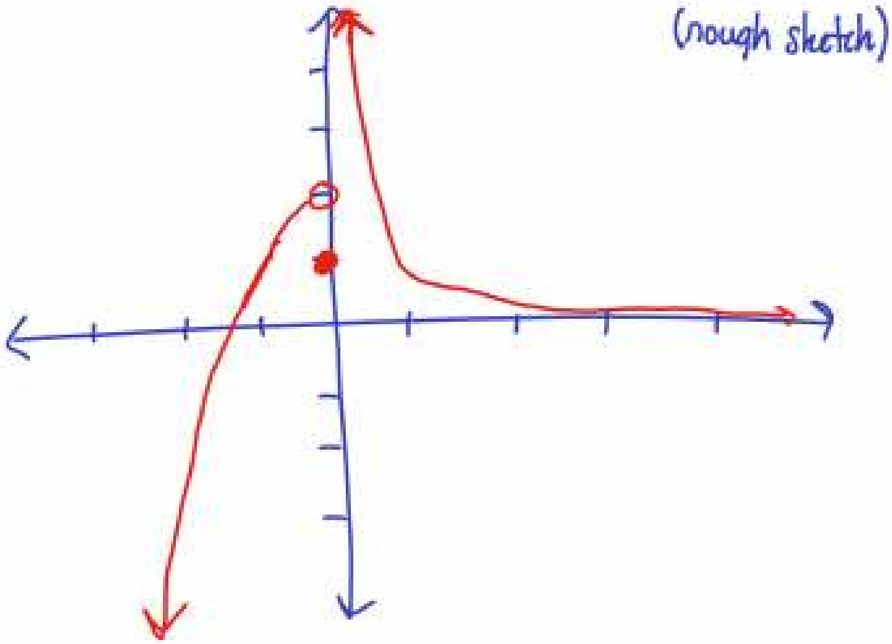
(d) Is f continuous at 2? Explain you answer using the definition of continuity.

No, the limit $\lim_{x\to 2} f(x)$ does not exist due to a jump ($\lim_{x\to 2^-} f(x)=-1$, but $\lim_{x\to 2^+} f(x)=3$), so the function is discontinuous there.

3. [20 points] Consider the function defined by

$$g(x) = \begin{cases} 1/x & x > 0 \\ 1 & x = 0 \\ 2 - x^2 & x < 0. \end{cases}$$

(a) Draw the graph of g.



(b) Use the graph of part (a) to find $\lim_{x\to 0^+} g(x)$, $\lim_{x\to 0^-} g(x)$, $\lim_{x\to 0} g(x)$ and $g(0)$.

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \left(\frac{1}{x}\right) = \frac{1}{0^+} = \boxed{+\infty}$$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (2-x^2) = \boxed{2}$$

$\lim_{x\to 0} g(x)$ $\boxed{\text{DNE}}$ (one-sided limits differ)

$$g(0) = \boxed{1}$$

4. [15 points] Let

$$f(x) = \frac{x+1}{x+2} \quad \text{and} \quad g(x) = \frac{1-x}{1+x}.$$

Simplify $f(g(x))$ as much as possible.

$$f(g(x)) = \frac{g(x)+1}{g(x)+2}$$

$$= \frac{\frac{1-x}{1+x}+1}{\frac{1-x}{1+x}+2}$$

$$= \frac{\frac{1-x+(1+x)}{1+x}}{\frac{1-x+2(1+x)}{(1+x)}}$$

$$= \frac{1-x+1+x}{\cancel{1+x}} \cdot \frac{\cancel{1+x}}{1-x+2+2x}$$

$$= \boxed{\frac{2}{x+3}}$$

5. [15 points] Suppose we know the limits

$$\lim_{x\to 2} f(x) = 4, \quad \lim_{x\to 2} g(x) = 3, \quad \lim_{x\to 2} h(x) = 0.$$

(a) What do the limit laws say about $\lim_{x\to 2} \frac{f(x)}{g(x)}$?

It is $\frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)} = \boxed{\frac{4}{3}}$

(b) What do the limit laws say about $\lim_{x\to 2} \frac{h(x)}{g(x)}$?

It is $\frac{\lim_{x\to 2} h(x)}{\lim_{x\to 2} g(x)} = \frac{0}{3} = \boxed{0}$

(c) What do the limit laws say about $\cancel{\lim_{x\to 2} \frac{f(x)}{h(x)}}$? $\lim_{x\to 2} \frac{g(x)}{h(x)}$

Nothing, since $\lim_{x\to 2} h(x) = 0$.

6. [8 points] Find the equation of the line perpendicular to the line $2x + 5y = 10$ that goes through the point $(-\frac{1}{2}, 2)$.

$2x+5y=10 \iff y = \frac{1}{5}(10-2x)$

$= -\frac{2}{5}x + 2$ (in slope-intercept form)

so the slope of a perpendicular line is $-\frac{1}{-2/5} = \frac{5}{2}$

Point-slope form: $(y-2) = \frac{5}{2}(x+\frac{1}{2})$

$\iff y = \frac{5}{2}x + \frac{5}{4} + 2 \iff \boxed{y = \frac{5}{2}x + \frac{13}{4}}$

7. [5 points (bonus)] Let $f(x) = 1 - x^2$. Compute

$$\lim_{h\to 0} \frac{f\left(\frac{1}{x+h}\right) - f\left(\frac{1}{x}\right)}{h}.$$

$$\lim_{h\to 0} \frac{\left(1-\left(\frac{1}{x+h}\right)^2\right) - \left(1-\left(\frac{1}{x}\right)^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{-\frac{1}{(x+h)^2} + \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{-x^2+(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{-x^2+(x+h)^2}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{-x^2+(x^2+2xh+h^2)}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}x^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{2x+h}{x^2(x+h)^2} = \frac{2x+0}{x^2\cdot(x+0)^2} = \frac{2x}{x^4}$$

$$= \boxed{2/x^3}$$