



*Amherst College*  
*Department of Mathematics and Statistics*

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MATH 105

MIDTERM 1

FALL 2018

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NAME: Solutions

**Read This First!**

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

**Grading - For Instructor Use Only**

Question:	1	2	3	4	5	6	Total
Points:	18	16	5	14	7	0	60
Score:							

1. [18 points] Evaluate the following limits. Answer either as a specific value,  $+\infty$ ,  $-\infty$ , or "DNE."

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow 4} \frac{x^2 - x - 12}{x^2 - 3x - 4} &= \lim_{x \rightarrow 4} \frac{(x-4)(x+3)}{(x-4)(x+1)} \\ &= \frac{4+3}{4+1} = \boxed{7/5} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 3^-} \frac{x-5}{x-3} &= \frac{3-5}{0^-} = \frac{-2}{0^-} = \boxed{+\infty} \end{aligned}$$

*Handwritten notes:*  
 x < 3 means x - 3 < 0  
 comp 4-1

$$\begin{aligned} \text{(c)} \quad \lim_{x \rightarrow -1} \frac{x^2 - 1}{\sqrt{x+5} - 2} \cdot \frac{\sqrt{x+5} + 2}{\sqrt{x+5} + 2} &= \lim_{x \rightarrow -1} \frac{(x+1)(x-1)(\sqrt{x+5} + 2)}{(x+5) - 4} \\ &= \lim_{x \rightarrow -1} \frac{(x+1)(x-1)(\sqrt{x+5} + 2)}{x+1} = (-2) \cdot (\sqrt{4} + 2) \\ &= \boxed{-8} \end{aligned}$$

$$(d) \lim_{x \rightarrow 2} \frac{x^2 + 2x - 7}{x^2 - 9} = \frac{4 + 4 - 7}{4 - 9} = \frac{1}{-5}$$

$$= \boxed{-1/5}$$

$$(e) \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{|x - 1|} \quad \text{DSP gives } 0/0$$

Consider the one-sided limits in order to simplify further:

$$\begin{aligned} \Rightarrow |x-1| = -(x-1) \quad & \lim_{x \rightarrow 1^-} \frac{x^2 - 3x + 2}{|x-1|} = \lim_{x \rightarrow 1^-} \frac{(x-1)(x-2)}{-(x-1)} = \frac{-1}{-1} = 1 \\ \Rightarrow |x-1| = (x-1) \quad & \lim_{x \rightarrow 1^+} \frac{x^2 - 3x + 2}{|x-1|} = \lim_{x \rightarrow 1^+} \frac{(x-1)(x-2)}{(x-1)} = \frac{-1}{1} = -1 \end{aligned}$$

They disagree, so the overall limit  $\boxed{\text{DNE}}$

$$(f) \lim_{x \rightarrow 1^+} \frac{x^2 + 4x + 3}{x^2 - 4x + 3} \quad \text{DSP gives } \frac{8}{0}; \text{ do sign analysis.}$$

$$= \lim_{x \rightarrow 1^+} \frac{(x+1)(x+3)}{(x-1)(x-3)} = \frac{2 \cdot 4}{0^+ \cdot (-2)} = \boxed{-\infty}$$

2. [16 points] Consider the following piecewise function.

$$f(x) = \begin{cases} 4 & x \leq -2 \\ 4 - x^2 & -2 < x \leq 0 \\ 1 + \frac{1}{x} & 0 < x < 1 \\ 3 - x & x \geq 1 \end{cases}$$

(a) Determine all discontinuities of this function.

The individual pieces are all continuous, except  $1 + \frac{1}{x}$ , which is discontinuous at  $x=0$ , but this is outside the interval where it is used.

So we need only check the break-points:  $x=-2, 0, \& 1$ .

$x=-2$

$$\text{LHL: } \lim_{x \rightarrow (-2)^-} f(x) = \lim_{x \rightarrow (-2)^-} (4) = 4$$

$$\text{RHL: } \lim_{x \rightarrow (-2)^+} f(x) = \lim_{x \rightarrow (-2)^+} (4 - x^2) = 4 - (-2)^2 = 0$$

These differ, so there's a jump discontinuity at  $x=-2$ .

$x=0$

$$\text{LHL: } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (4 - x^2) = 4 - 0^2 = 4$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (1 + \frac{1}{x}) = 1 + \frac{1}{0^+} = +\infty$$

One of these is infinite, so there's an infinite discontinuity at  $x=0$ .

$x=1$

$$\text{LHL: } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (1 + \frac{1}{x}) = 1 + \frac{1}{1} = 2$$

$$\text{RHL: } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (3 - x) = 3 - 1 = 2$$

These agree & equal  $f(1) = 3 - 1 = 2$ , so  $f(x)$  is continuous here.

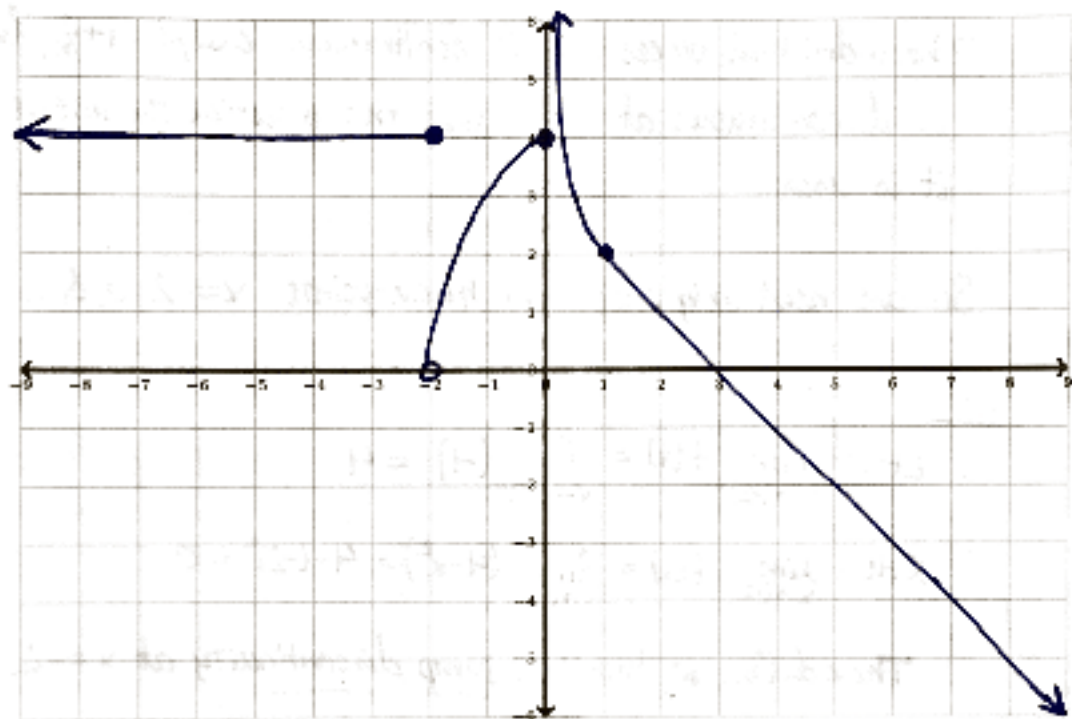
(continued on reverse)

Discontinuities at  $\boxed{x=-2 \text{ and } x=0}$ .

The definition of  $f(x)$  is reproduced below for convenience.

$$f(x) = \begin{cases} 4 & x \leq -2 \\ 4 - x^2 & -2 < x \leq 0 \\ 1 + \frac{1}{x} & 0 < x < 1 \\ 3 - x & x \geq 1 \end{cases}$$

(b) Sketch the graph of this function on the axes below.



3. [5 points] Find an equation for a line that is parallel to the line  $3x+4y=24$  and passes through the point  $(1,2)$ .

$$3x+4y=24$$

$$\Leftrightarrow 4y = -3x + 24$$

$$\Leftrightarrow y = -\frac{3}{4}x + 6$$

which has slope  $-3/4$ .

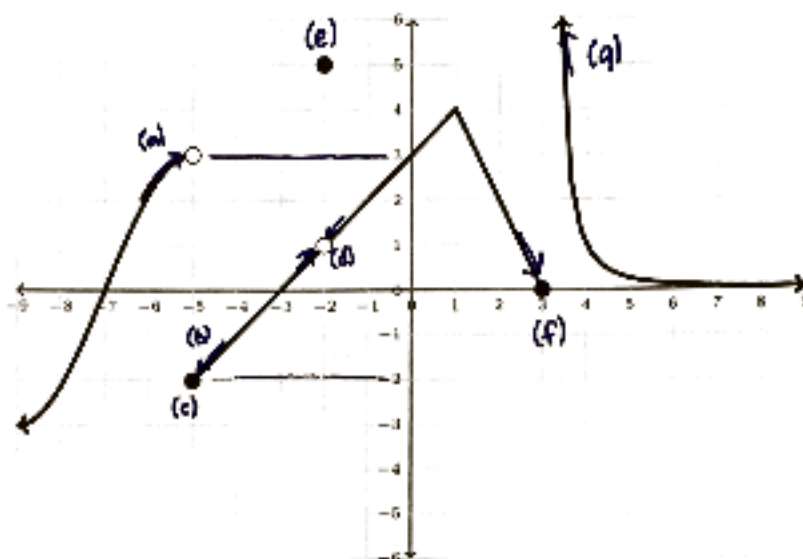
Parallel lines all have slope  $-3/4$ . So the desired line, in point-slope form, is

$$(y-2) = -\frac{3}{4}(x-1)$$

ie.  $y-2 = -\frac{3}{4}x + \frac{3}{4}$

$$\Leftrightarrow \boxed{y = -\frac{3}{4}x + \frac{11}{4}}$$

4. [14 points] Shown below is the graph of a function  $f(x)$ . Determine the following limits of function values. If a value is  $+\infty$  or  $-\infty$ , state this. If a value does not exist, answer "DNE."



(a)  $\lim_{x \rightarrow (-5)^-} f(x)$

3

(e)  $f(-2)$

5

(b)  $\lim_{x \rightarrow (-5)^+} f(x)$

-2

(f)  $\lim_{x \rightarrow 3^-} f(x)$

0

(c)  $f(-5)$

-2

(g)  $\lim_{x \rightarrow 3^+} f(x)$

$+\infty$

(d)  $\lim_{x \rightarrow (-2)} f(x)$

1

- (h) Identify all values of  $x$  where  $f(x)$  is discontinuous.

$x = -5$  (jump)  
 $x = -2$  (removable)  
 $x = 3$  (infinite)

5. [7 points] Define  $f(x)$  and  $g(x)$  as follows.

$$f(x) = \frac{2+x}{1-x}$$

$$g(x) = \frac{3x+1}{x-2}$$

Compute and simplify  $f(g(x))$  as much as possible.

$$\begin{aligned} f(g(x)) &= \frac{2+g(x)}{1-g(x)} \\ &= \frac{2 + \frac{3x+1}{x-2}}{1 - \frac{3x+1}{x-2}} \\ &= \frac{\frac{2(x-2) + (3x+1)}{x-2}}{\frac{(x-2) - (3x+1)}{x-2}} \\ &= \frac{2x-4+3x+1}{x-2} \cdot \frac{x-2}{x-2-3x-1} \\ &= \frac{5x-3}{-2x-3} \\ &= \boxed{-\frac{5x-3}{2x+3}} \end{aligned}$$

6. [3 points (bonus)] Let  $f(x) = 2 - \sqrt{3x+1}$ . Compute

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Note that the answer will be in terms of  $x$ .

$$\lim_{h \rightarrow 0} \frac{(2 - \sqrt{3(x+h)+1}) - (2 - \sqrt{3x+1})}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2 - \sqrt{3x+3h+1} - 2 + \sqrt{3x+1}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{3x+1} - \sqrt{3x+3h+1}}{h} \cdot \frac{\sqrt{3x+1} + \sqrt{3x+3h+1}}{\sqrt{3x+1} + \sqrt{3x+3h+1}}$$

$$= \lim_{h \rightarrow 0} \frac{(3x+1) - (3x+3h+1)}{h(\sqrt{3x+1} + \sqrt{3x+3h+1})}$$

$$= \lim_{h \rightarrow 0} \frac{-3h}{h(\sqrt{3x+1} + \sqrt{3x+3h+1})}$$

$$= \frac{-3}{(\sqrt{3x+1} + \sqrt{3x+1})}$$

$$= \boxed{-\frac{3}{2\sqrt{3x+1}}}$$